Development, Fertility and Childbearing Age: A Unified Growth Theory

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Abstract

During the last two centuries, fertility has exhibited, in industrialized economies, two distinct trends: the cohort total fertility rate follows a decreasing pattern, while the cohort average age at motherhood exhibits a U-shaped pattern. This paper proposes a unified growth theory aimed at rationalizing those two demographic stylized facts. We develop a three-period OLG model with two periods of fertility, and show how a traditional economy, where individuals do not invest in higher education, and where income rises push towards advancing births, can progressively converge towards a modern economy, where individuals invest in higher education, and where income rises encourage postponing births. Our findings are illustrated numerically by replicating the dynamics of the *quantum* and the *tempo* of births for Swedish cohorts born between 1876 and 1966.

Keywords: fertility, childbearing age, births postponement, human capital, regime shift.

JEL codes: J11, J13, O12.

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1 Introduction

In the 20th century, growth theorists paid particular attention to interactions between, on the one hand, the production of goods, and, on the other hand, fertility behavior, that is, the production of men. When studying those interactions, they have mainly focused on one aspect of fertility: the number or *quantum* of births. From that perspective, the key stylized fact to be explained is the declining trend in fertility.[1] That decline is illustrated on Figure 1, which shows the completed total fertility rate (TFR) for cohorts of women aged 40 in industrialized countries. That fertility decline was explained through various channels, such as the rise in the opportunity costs of children (Barro and Becker 1989), a shift from investment in quantity towards quality (education) caused by lower infant mortality (Erhlich and Lui 1991), a rise in women's relative wages (Galor and Weil 1996), and the rise of contraception (Strulik 2016).

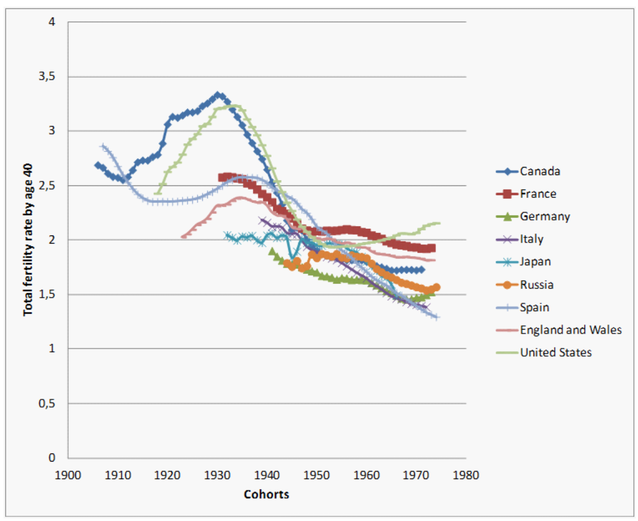

Figure 1: Completed cohort fertility by age 40
(source: Human Fertility Database)

Although those models cast substantial light on the interactions between fertility and development, their exclusive emphasis on the *quantum* of births leaves aside another important aspect of fertility, which has a strong impact on economic development: the timing or *tempo* of births. Studying the *tempo* of births - and not only the *quantum* of births - matters for understanding long-run economic development, because of two distinct reasons.

First, theoretical papers, such as Happel et al (1984), Cigno and Ermisch (1989) and Gustafsson (2001), studied, at the microeconomic level, the mech-

[1] Note that, although the long-run trend of the TFR is decreasing, the TFR can nonetheless exhibit significant short-run fluctuations around that trend, as shown on Figure 1.

anisms by which the birth timing decision is related to education and labor supply decisions. A lower wage early in the career reduces the opportunity cost of an early birth, and pushes towards advancing births (substitution effect), but limits also the purchasing power, which pushes towards postponing births (income effect). Moreover, investing in education raises the purchasing power in the future (which pushes towards postponing births), but, at the same time, raises also the opportunity cost of future children (which pushes towards advancing births). Those strong interactions between birth timing, education and labor decisions at the temporary equilibrium motivate the study, in a dynamic model, of how development affects - and is affected by - birth timing.

Second, there is also strong empirical evidence supporting the existence of complex, multiple interactions between the *tempo* of births and various economic variables, with causal relations going in both directions. Demographic studies show that the *tempo* of births is strongly correlated with the education level, which affects the human capital accumulation process (see Smallwood, 2002, Lappegard and Ronsen 2005, Robert-Bobée et al 2006). Moreover, several works, such as Schultz (1985), Heckman and Walker (1990) and Tasiran (1995), show that a rise in women's wages tends to favor a postponement of births.[2] There is also evidence that the wage level is affected by the timing of births (see Miller 1989, Joshi 1990, 1998, Dankmeyer 1996).

The timing of births has varied significantly during the 20th century, as illustrated on Figure 2, which shows the cohort mean age at birth by age 40.[3] Whereas the patterns differ across countries, Figure 2 reveals an important stylized fact: the average age at motherhood exhibits, across cohorts, a U-shaped pattern. The average age at motherhood has been first decreasing for cohorts born before 1940/1950, and, then, has been increasing for later cohorts.[4]

The U-shaped cohort mean age at birth raises several questions. A first question concerns the economic causes and consequences of that non-monotonic pattern. How can one explain that economic development is associated first with advancing births, and, then, with postponing births? How can one relate this stylized fact with income and substitution effects, and with the education decision? Another key question concerns the relation between the dynamics of the *quantum* of births (Figure 1) and the *tempo* of births (Figure 2). Why is it the case that, at a time of strong fertility decline, cohorts tended to advance births, and, then, tended to postpone births once total fertility was stabilized?

Exploring the relationship between the *quantum* and the *tempo* of births is most crucial for understanding long-run development, and, in particular, the conditions allowing for an economic take-off. Economic development is often associated with the decline of fertility, but, at the same time, as long as the age

[2] Those studies focused on Sweden. Similar results were shown for Japan (Ermish and Ogawa 1994), for Canada (Merrigan and Saint-Pierre 1998), and for the UK (Joshi 2002).

[3] The cohort mean age at birth by age 40 is computed for *all* births combined (and not only for first births). As such, it is likely to depend on the *quantum* of births (women having more children being likely to be older, on average, when giving birth to a child). This paper focuses precisely on this relation between the *tempo* of births and the *quantum* of births.

[4] Note that the timing of the reversal varies across countries. For instance, in Russia, the reversal of the mean age at motherhood occurred for cohorts born in the late 1960s.

at motherhood remains low, this may prevent investments in higher education, and, hence, prevent the emergence of a sustainable economic take-off. Therefore, in order to understand the emergence of the take-off, one must go beyond the study of the quantity of births, to examine also the timing of births.

The goal of this paper is precisely to cast some light on the relation between the *quantum* of births, the *tempo* of births, and economic development. For that purpose, we propose to adopt a unified growth approach. As pioneered by Galor (see Galor and Moav, 2002, Galor, 2010), the unified growth approach pays a particular attention to the relation between quantitative changes (i.e. changes in numbers) and qualitative changes (i.e. changes in the form of relations between variables). Qualitative changes are here studied by means of regime shifts, which are achieved as the economy develops, and which cause major changes in the relations between fundamental variables.[5] As such, the unified growth approach is most adequate to study the U-shaped pattern exhibited by the *tempo* of births.

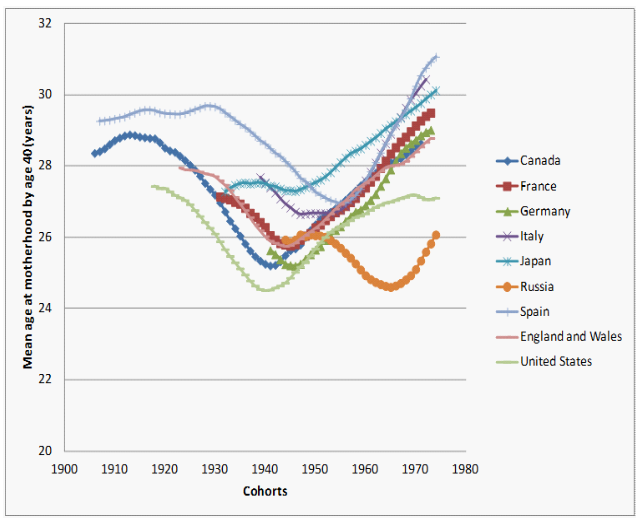

Figure 2: Cohort mean age at birth by age 40 (Source: Human Fertility Database)

For that purpose, this paper develops a three-period overlapping generations (OLG) model with lifecycle fertility, that is, with two fertility periods (instead of one as usually assumed). Individuals can thus decide not only the *quantum* of births, but, also, how they allocate those births along their lifecycle, that is, the *tempo* of births. In order to study the interactions between birth timing and education, we also assume that individuals can choose how much they invest in their education when being young, which will affect their future productivity (and, hence, the opportunity cost of late births).

[5]Recent works in that approach include de la Croix and Licandro (2013), de la Croix and Mariani (2015), Lindner and Strulik (2015) and Strulik (2016).

Anticipating our results, we show that, depending on the prevailing level of human capital, the temporary equilibrium takes three distinct forms, which corresponds to three distinct regimes. In the first regime, individuals do not invest in education, and rises in income push them towards advancing births. In the second regime, individuals start investing in education, but education remains so low that income rises still make individuals advance births. Then, once human capital is sufficiently high, the economy enters a third regime, where income growth favors postponing births.

Our dynamic lifecycle fertility model can thus rationalize both the decrease in fertility and the U-shaped pattern of the mean age at birth. That rationalization of the non-monotonic relation between development and birth timing is achieved by means of regime shifts as the economy develops, without having to rely on exogenous shocks. Besides this analytical finding, we also explore the capacity of our model to replicate numerically the dynamics of the *quantum* and the *tempo* of births. Taking the case of Swedish women cohort born between 1876 and 1966, we show that our model can reasonably fit the patterns of the cohort total fertility rate and the cohort mean age at motherhood.

Our paper is related to several branches of the literature. First of all, it complements microeconomic studies of birth timing, such as Happel et al (1984), Cigno and Ermisch (1989) and Gustafsson (2001), which examine birth timing decisions in a static setting, whereas we propose to draw the corollaries of those decisions for long-run dynamics. Second, we also complement recent works focusing on the relation between birth timing and long-run development, such as, in continuous time, d'Albis et al (2010), and, in discrete time, Momota and Horii (2013), and Pestieau and Ponthiere (2014, 2015). Those papers examined the relation between, on the one hand, physical capital accumulation, and, on the other hand, the *quantum* and *tempo* of births. We complement those papers by paying attention to the interactions with the education decision and the human capital accumulation process. Moreover, another extra value of our paper lies in the fact that it adopts a unified growth approach, where regime shifts are used to rationalize the non-monotonic pattern exhibited by the *tempo* of births.

The rest of the paper is organized as follows. The model is presented in Section 2. The temporary equilibrium is studied in Section 3, which characterizes the three possible regimes. Section 4 explores the long-run dynamics of our economy. Section 5 illustrates our findings numerically, by focusing on the case of Swedish women cohorts born between 1876 and 1966. Section 6 concludes.

2 The model

Let us consider a three-period OLG model with lifecycle fertility. Time goes from 0 to $+\infty$. Each period has a unitary length. Period 1 is childhood, during which the child is raised by the parents, and does not make any decision. Period 2 is early adulthood, during which individuals work, consume, have n_t children and invest in higher education. Period 3 is mature adulthood, during which individuals work, consume, and can complete their fertility by having m_{t+1}

children. Figure 3 shows the formal structure of the model.

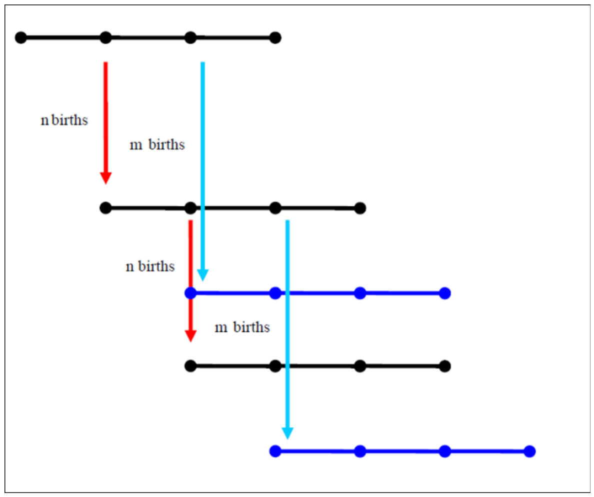

Figure 3: The lifecycle fertility model.

Production Production involves labor and human capital. The output of an agent at time t, denoted by y_t, is equal to:

$$y_t = h_t \ell_t \tag{1}$$

where h_t is the human capital of the agent at time t, while ℓ_t is the labor time.

Human capital accumulation When becoming a young adult at time t, each agent is endowed with a human capital level $h_t > 0$.[6] This human capital level determines the marginal productivity of his labor when being a young adult.

The young adult can invest in higher education, in such a way as to increase his human capital stock at the next period. Human capital accumulates according to the law:

$$h_{t+1} = (v + e_t)\, h_t \tag{2}$$

where e_t denotes the level of effort/investment in higher education, while v is an accumulation parameter, which determines the rate at which human capital accumulates in the absence of higher education (i.e. when $e_t = 0$). We assume here, for analytical tractability, that education takes the form of a non-monetary, non-temporal, physical effort, which can take any positive value.

[6] This is true for all individuals born in $t-1$, whaterver these are themselves early or late children. This uniformity of endowments across children allows us to keep a representative agent structure, unlike in Pestieau and Ponthiere (2016), where early and late children, who differ in terms of time constraints, are studied as two distinct populations.

Throughout this paper, we will also suppose that $v > 1$, that is, that even in the absence of higher education, individuals can become more productive over time. Human skills can improve despite the absence of higher education, because either of a standard learning by doing mechanism, or of an exogenous technological progress raising the productivity of labor.

Budget constraints It is assumed that raising a child has a time cost $q \in]0,1[$. That cost is supposed to be the same for early and late children. Thus, assuming that there is no savings, so that each agent consumes what he produces at each period of life, the budget constraint at early adulthood is:[7]

$$c_t = h_t \left(1 - q n_t\right) \tag{3}$$

where c_t denotes consumption at early adulthood for a young adult at time t.

The budget constraint at mature adulthood is:

$$d_{t+1} = h_{t+1} \left(1 - q m_{t+1}\right) \tag{4}$$

where d_{t+1} denotes consumption at mature adulthood for a mature adult at time $t+1$.

Preferences Individuals derive some utility from consumption and from having children. They also derive disutility from investing efforts in higher education.

Individuals are endowed with preferences having a log linear form:

$$\alpha \log\left(c_t + \delta\right) - \sigma \log\left(e_t + \eta\right) + \beta \log(d_{t+1} + \varepsilon) + \gamma \log(n_t) + \rho \log(m_{t+1}) \tag{5}$$

where $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$, $\sigma > 0$, $\varepsilon > 0$, $\eta > 0$ and $\rho > 0$ are preference parameters. α and β capture the weight assigned to consumption during the life. σ captures the disutility of higher education efforts. γ (resp. ρ) captures the taste for early (resp. late) fertility. Parameters δ, η and ε are used to allow for the emergence of various types of temporary equilibria, exhibiting either interior or corner solutions for the choice variables.

An important feature of the above utility function is that it exhibits limited substitutability between early births and late births. The main justification for this feature is that, in the same way as there is no perfect substitutability for the allocation of consumption across time periods, there is no perfect substitutability for the allocation of births across time periods.

3 The temporary equilibrium

At the beginning of early adulthood, individuals choose the higher education effort e_t, the number of early children n_t, and the number of late children m_{t+1},

[7] It is assumed here that investing in higher education does not involve either monetary costs or time costs. Higher education is here treated as a kind of effort, which can raise future human capital, but at the cost of some disutility at early adulthood. See below on this.

in such a way as to maximize their lifetime welfare, subject to their budget constraints. The problem can be written as:

$$\begin{aligned}\max_{e_t,n_t,m_{t+1}} \quad & \alpha \log\left(h_t(1-n_tq)+\delta\right) - \sigma\log\left(e_t+\eta\right) \\ & +\beta\log((v+e_t)h_t(1-m_{t+1}q)+\varepsilon) + \gamma\log(n_t) + \rho\log(m_{t+1})\end{aligned}$$

The first-order conditions (FOCs) for, respectively, optimal interior levels of education e_t, early fertility n_t and late fertility m_{t+1}, are:

$$\frac{\sigma}{(e_t+\eta)} = \frac{\beta h_t(1-m_{t+1}q)}{(v+e_t)h_t(1-m_{t+1}q)+\varepsilon} \tag{6}$$

$$\frac{\alpha h_t q}{h_t(1-n_tq)+\delta} = \frac{\gamma}{n_t} \tag{7}$$

$$\frac{\beta(v+e_t)h_t q}{(v+e_t)h_t(1-m_{t+1}q)+\varepsilon} = \frac{\rho}{m_{t+1}} \tag{8}$$

The first FOC characterizes the optimal interior level of higher education effort. It states that the optimal higher education effort equalizes the marginal disutility of education effort (LHS) and the marginal utility gain from extra consumption at mature adulthood thanks to education (RHS).

The second FOC, which characterizes the optimal early fertility level, equalizes the marginal utility gain from early fertility (RHS) with the marginal utility loss from early fertility (LHS). Note that, since $\delta > 0$, the marginal utility loss of early births depends on the prevailing level of human capital. This would not be the case under $\delta = 0$, since in that case the income and substitution effects would cancel each other, making fertility independent from human capital.

Finally, the third FOC characterizes the optimal late fertility level. It equalizes the marginal utility gain from late births (RHS) with the marginal utility loss from late births (LHS). Note that, under $\varepsilon > 0$, the latter depends on the education level, which affects both the purchasing power at mature adulthood and the opportunity cost of late births.

Whereas the 3 FOCs characterize a temporary equilibrium with interior levels for education e_t, early fertility n_t and late fertility m_{t+1}, such an interior temporary equilibrium is not necessarily reached, depending on the level of human capital at time t. Proposition 1 summarizes the three different regimes. Those regimes differ not only in terms of the levels of education e_t and total fertility $(n_t + m_{t+1})$, but, also, in terms of the timing of births, which is here measured by the ratio early births / late births, $R_t \equiv \frac{n_t}{m_{t+1}}$.

Proposition 1 *Define the function:* $e(h_t) \equiv \frac{-[h_t\Omega+\varphi]+\sqrt[2]{\Delta(h_t)}}{2h_t\omega}$, *where* $\Delta(h_t) \equiv [h_t\Omega+\varphi]^2 + 4h_t\omega v\left(\eta\beta - v\sigma\right)\left(h_t - \bar{h}\right)$, *and where* $\Omega \equiv 2\sigma v - (v+\eta)\beta$, $\varphi \equiv \varepsilon\left(\sigma+\rho\right)$, $\omega \equiv (\sigma-\beta)$ *and* $\bar{h} \equiv \frac{\varepsilon(\sigma v+\rho\eta)}{v(\beta\eta-\sigma v)}$.

Define $\tilde{h}$ *as the solution to:* $e(h) + (h+\delta)\,e'(h) = \frac{\delta}{\varepsilon}(v+e(h))^2 - v$.

There exist sets of values for the parameters such that the level of human capital defines three regimes:

- ***Regime I***: *if* $h_t < \bar{h}$, *then:*

$$\begin{aligned}
e_t^I &= 0 \\
n_t^I &= \frac{\gamma\left(h_t+\delta\right)}{h_t q\left(\alpha+\gamma\right)} > 0 \\
m_{t+1}^I &= \frac{\rho(v h_t+\varepsilon)}{v h_t q\left(\beta+\rho\right)} > 0 \\
R_t^I &= \frac{\gamma\left(h_t+\delta\right) v\left(\beta+\rho\right)}{\left(\alpha+\gamma\right)\rho(v h_t+\varepsilon)} > 0 \\
\frac{\partial e_t^I}{\partial h_t} &= 0, \frac{\partial n_t^I}{\partial h_t} < 0, \frac{\partial m_{t+1}^I}{\partial h_t} < 0, \frac{\partial R_t^I}{\partial h_t} > 0
\end{aligned}$$

- ***Regime II***: *if* $\bar{h} < h_t < \tilde{h}$, *then:*

$$\begin{aligned}
e_t^{II} &= e(h_t) > e_t^I \\
n_t^{II} &= \frac{\gamma\left(h_t+\delta\right)}{h_t q\left(\alpha+\gamma\right)} < n_t^I \\
m_{t+1}^{II} &= \frac{\rho((v+e(h_t))h_t+\varepsilon)}{(v+e(h_t))h_t q\left(\beta+\rho\right)} < m_{t+1}^I \\
R_t^{II} &= \frac{\gamma\left(h_t+\delta\right)\left(v+e(h_t)\right)\left(\beta+\rho\right)}{\left(\alpha+\gamma\right)\rho((v+e(h_t))h_t+\varepsilon)} > R_t^I \\
\frac{\partial e_t^{II}}{\partial h_t} &> 0, \frac{\partial n_t^{II}}{\partial h_t} < 0, \frac{\partial m_{t+1}^{II}}{\partial h_t} < 0, \frac{\partial R_t^{II}}{\partial h_t} > 0
\end{aligned}$$

- ***Regime III:*** *if* $\tilde{h} < h_t$, *then:*

$$\begin{aligned}
e_t^{III} &= e(h_t) > e_t^{II} \\
n_t^{III} &= \frac{\gamma\left(h_t+\delta\right)}{h_t q\left(\alpha+\gamma\right)} < n_t^{II} \\
m_{t+1}^{III} &= \frac{\rho((v+e(h_t))h_t+\varepsilon)}{(v+e(h_t))h_t q\left(\beta+\rho\right)} < m_{t+1}^{II} \\
R_t^{III} &= \frac{\gamma\left(h_t+\delta\right)\left(v+e(h_t)\right)\left(\beta+\rho\right)}{\left(\alpha+\gamma\right)\rho((v+e(h_t))h_t+\varepsilon)} < R_t^{II} \\
\frac{\partial e_t^{III}}{\partial h_t} &> 0, \frac{\partial n_t^{III}}{\partial h_t} < 0, \frac{\partial m_{t+1}^{III}}{\partial h_t} < 0; \frac{\partial R_t^{III}}{\partial h_t} < 0
\end{aligned}$$

Proof. See the Appendix. ■

Proposition 1 states that, depending on the prevailing level of the human capital stock, there can exist, in our economy, three distinct regimes.

Under the first regime, human capital is low, and there is no higher education. The reason is that the marginal return of investing in education is, given the low human capital stock, too low with respect to its marginal disutility, which makes investment in higher education not worthy. Quite interestingly, in that regime, a rise in human capital reduces both early and late fertility, but pushes parents towards having a larger proportion of early births (i.e. human capital raises the ratio R_t, that is, the ratio early births over late births). Thus, in Regime I, the rise in income pushes towards advancing births. The intuition behind that result is that, under the assumption $\varepsilon > \delta v$, we have that, under a zero education, the marginal utility loss from foregone consumption due to early births is, *ceteris paribus*, less increasing with h_t than the marginal utility loss from foregone consumption due to late births.[8]

In the second regime, higher education is now strictly positive, since the human capital is sufficiently large, so that the marginal return from investing in education is sufficiently high so as to counterbalance the marginal disutility of investing in education. In that second regime, education is increasing in human capital. Note also that, in Regime II, fertility at both periods is now lower than under Regime I. But the accumulation of human capital - and, hence, the rise in income - has still the consequence of advancing births.

In the third regime, fertility is even lower than in the two previous regimes, but there is an important qualitative change. Whereas the rise in income used to push towards advancing births in Regimes I and II, this is no longer the case in Regime III, where the rise in income pushes towards postponing births (that is, human capital reduces the ratio R_t). This third regime coincides with what could be called the modern regime, where the decline in fertility is associated with births postponement. Note that, whereas Regime III may seem, at first glance, quite similar to Regime II, these differ significantly on quantitative and qualitative aspects. From a quantitative perspective, Regime III is characterized by lower fertility, and by a higher average age at motherhood. From a qualitative perspective, Regime III is characterized by an inversion, with respect to Regime II, of the relation between income growth and birth timing.

In sum, Proposition 1 shows that, depending on the prevailing level of human capital, the economy can be characterized by three distinct regimes, where the relations between income, the *quantum* at birth and the *tempo* of births vary. Each of those regimes can be regarded as a particular stage of development. The next section aims at describing how the economy shifts from one regime to the next as human capital accumulates.

4 Long-run development

In order to study how our economy evolves over time, this section will first consider the dynamics of human capital accumulation. In a second stage, we will study the dynamics of the quantity and the timing of births.

[8] To better see this, note that, if δ were equal to 0, the marginal utility loss from foregone consumption due to early births would be invariant to h_t.

4.1 Human capital growth

Starting from an initial human capital level $h_0 < \bar{h}$, the economy is initially in Regime I. Under that regime, education investment is equal to zero and human capital grows at an exogenous, positive rate v. Once the human capital stock crosses the threshold $\bar{h}$, the economy enters Regime II, under which education effort becomes positive. Hence the growth rate is now:

$$g_{t+1}^{II} = v + e_t^{II} > g_{t+1}^{I} = v$$

Once the human capital stock reaches a second threshold, equal to $\tilde{h} > \bar{h}$, the economy enters into Regime III, where the human capital stock grows at a rate:

$$g_{t+1}^{III} \equiv \frac{h_{t+1}}{h_t} = v + e_t^{III}$$

Given that education effort is increasing with human capital (so that $e_t^{III} > e_t^{II}$), we obtain, when comparing the growth rates of human capital, that:

$$g_{t+1}^{III} > g_{t+1}^{II} > g_{t+1}^{I} = v$$

Thus, once it has reached the modern regime (i.e. Regime III), the human capital stock grows even more quickly than under the two preceding regimes.

Note that, whereas human capital grows without limit in Regime III, we know, since education converges towards a finite positive level (see Proof of Proposition 1 in the Appendix), that the growth rate of human capital converges asymptotically towards a positive level, equal to:

$$g_\infty = v + \lim_{h \to \infty} e(h_t) = v + \frac{-\Omega}{2\omega} + \sqrt[2]{\frac{2\Omega^2 + 8\omega v\,(\eta\beta - v\sigma)}{8\omega^2}} > 0$$

The following proposition summarizes our results.

Proposition 2

- *Under* ***Regime I****, the human capital stock grows at an exogenous constant rate* $v > 0$.

- *Under* ***Regime II*** *and* ***Regime III****, the human capital stock grows at a rate that is higher than* v*. That growth rate is increasing in education, which is itself increasing in* h_t*:*

$$g_{t+1}^{III} > g_{t+1}^{II} > g_{t+1}^{I} = v$$

- *The growth rate of human capital tends, in the long-run, towards the level:*

$$g_\infty = v + \frac{-\Omega}{2\omega} + \sqrt[2]{\frac{2\Omega^2 + 8\omega v\,(\eta\beta - v\sigma)}{8\omega^2}}$$

Proof. See above. ■

Our model thus predicts an acceleration of human capital accumulation as the economy develops. The first stage of development (i.e. Regime I) is characterized by lower human capital accumulation, since education efforts are limited at that stage. But when the economy enters Regime II, education efforts become positive, and growing with the stock of human capital, which makes the growth of human capital stock even stronger than in Regime II. Finally, once the economy reaches the second threshold, and enters Regime III, the growth rate of human capital is even larger than in Regime II. The reason is that, in Regime III, human capital accumulation pushes towards births postponement, which has here the impact of increasing even more the attractiveness of higher education investment, which in turn strengthens human capital accumulation.

4.2 Quantum and tempo of births

Besides the dynamics of human capital accumulation, one may also be interested in studying the evolution of total fertility and of the timing of births when the economy develops. For that purpose, we can, in our model, define the cohort total fertility rate (TFR) as:

$$TFR_t \equiv n_t + m_{t+1}$$

as well as the cohort mean age at birth (MAB) as:

$$MAB_t \equiv \frac{n_t \times 1 + m_{t+1} \times 2}{n_t + m_{t+1}}$$

where it is here assumed that individuals having children during a period have those children at the beginning of the period (and thus have age 1 for early births and age 2 for late births).

Again, we can consider, as a starting point, an economy with initial human capital $h_0 < \bar{h}$. In Regime I, human capital accumulates at a rate v. Given that both n_t and m_{t+1} are decreasing in human capital under either Regimes I, II or III, it is straightforward to deduce that the cohort TFR is decreasing as the economy develops:

$$TFR_t^I > TFR_t^{II} > TFR_t^{III}$$

Concerning the evolution of the cohort mean age at birth, one can notice that the MAB can be rewritten as:

$$MAB_t = \frac{R_t + 2}{R_t + 1}$$

The mean age at birth is thus decreasing in the ratio R_t. From Proposition 1, we know that the ratio R_t is, in Regimes I and II, increasing with human capital accumulation, so that we have:

$$MAB_t^I > MAB_t^{II}$$

However, once the economy has entered Regime III, the relation between human capital accumulation and birth timing is inverted, and the ratio R_t becomes decreasing with human capital, so that we know have:

$$MAB_t^{II} < MAB_t^{III}$$

Thus, as the economy develops, we can observe a monotonic decline in the *quantum* of births (TFR), and a non monotonic, U-shaped relation in the MAB.

Note also that, since education converges asymptotically towards a positive constant, it is also the case that both early and late fertility converges towards a constant in the long-run. These are equal to, respectively:

$$\begin{aligned} \lim_{h_t \to \infty} n_t &= \frac{\gamma}{q(\alpha+\gamma)} > 0 \\ \lim_{h_t \to \infty} m_{t+1} &= \frac{\rho}{q(\beta+\rho)} > 0 \end{aligned}$$

As a consequence, the TFR and MAB converge asymptotically towards, respectively:

$$\begin{aligned} TFR_\infty &= \frac{\gamma}{q(\alpha+\gamma)} + \frac{\rho}{q(\beta+\rho)} \\ MAB_\infty &= \frac{\gamma(\beta+\rho) + 2\rho(\alpha+\gamma)}{\gamma(\beta+\rho) + (\alpha+\gamma)\rho} \end{aligned}$$

The following proposition summarizes our results.

Proposition 3

- *Comparing the levels of the cohort TFR across regimes, we have a monotonic decline in TFR as the economy develops, and goes from **Regime I** to **Regimes II** and **III**:*

$$TFR_t^I > TFR_t^{II} > TFR_t^{III}$$

- *Comparing the levels of the cohort MAB across regimes, we have a non monotonic, U-shaped pattern for the MAB as the economy develops, and goes from **Regime I** to **Regimes II** and **III**:*

$$MAB_t^I > MAB_t^{II} < MAB_t^{III}$$

- *The cohort TFR and the cohort MAB converge asymptotically towards, respectively:*

$$\begin{aligned} TFR_\infty &= \frac{\gamma}{q(\alpha+\gamma)} + \frac{\rho}{q(\beta+\rho)} \\ MAB_\infty &= \frac{\gamma(\beta+\rho) + 2\rho(\alpha+\gamma)}{\gamma(\beta+\rho) + (\alpha+\gamma)\rho} \end{aligned}$$

Proof. See above. ■

In the light of Proposition 3, it is possible to rationalize the observed evolution of the *quantum* and the *tempo* of births as the economy develops. While the first stages of development are characterized by large quantity of children and high age at motherhood (i.e. high TFR and high MAB), the *quantum* of births decreases as the economy develops, whereas the timing of births follows a non-monotonic pattern: economic development first pushes towards advancing births (i.e. a lower MAB), and, then, towards postponing these (i.e. a higher MAB).

5 Numerical analysis

The previous Sections show that our model can, qualitatively, explain or rationalize the global patterns exhibited by the *quantum* and the *tempo* of births. One may want to go further in the replications, and wonder to what extent it is possible, by calibrating our model, to reproduce the TFR and MAB patterns for a real-world economy. This is the task of this Section.

For that purpose, we will consider here the case of Sweden, for which we have the longest time series of cohort TFR and cohort MAB in the Human Fertility Database. Figure 4 shows the patterns of the TFR at age 40 for cohorts born between 1876 and 1974, while Figure 5 shows the pattern of the MAB at age 40 for the same cohorts. As shown on Figure 4, there has been a substantial decline in the *quantum* of births across the periods considered, from about 3.2 children per women for the cohort born in 1876 towards less than 2 children per women for the cohort born in 1974. Note, however, that there has been some short-run fluctuations in the TFR, with some rebound for cohorts born in the 1900s, as well as a second rebound for those born in the 1930s. Concerning the evolution of the MAB (Figure 5), the curve exhibits a U-shaped pattern, as mentioned in Section 1. The MAB was quite high - around age 30 - for Swedish women born in 1876, but then strongly declined, to reach about 26 years for cohorts born in the 1940s. Then, there was a substantial births postponement for cohorts born after 1950. For cohorts born in the 1970s, the MAB is close to its level for cohorts born one century before. Note, however, that the MAB has also exhibited short-run fluctuations. In particular, cohorts born in the 1910s have exhibited a higher MAB than those born in the 1900s.

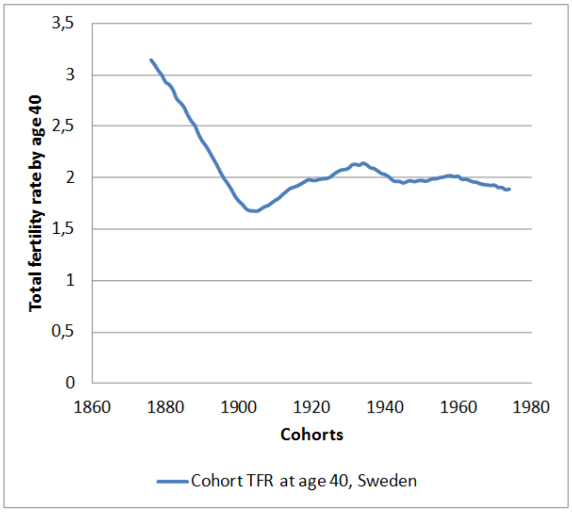

Figure 4: Cohort TFR at age 40 for Swedish cohorts 1876-1974 (source: Human Fertility Database).

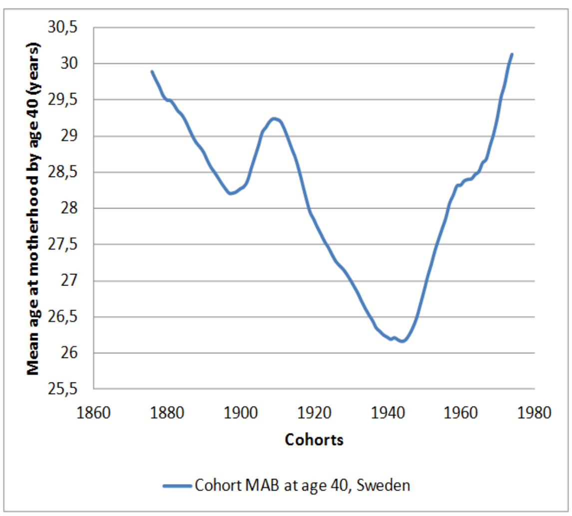

Figure 5: Cohort MAB by age 40 for Swedish cohorts 1876-1974 (Source: Human Fertility Database).

Figure 5 makes appear a major difficulty in trying to replicate quantitatively the observed pattern of the *tempo* of births by means of a discrete time OLG model with relatively long fertility periods (and thus with few coexisting cohorts). The difficulty lies in the fact that the timing of births has exhibited massive and quick changes in a very small number of (annual) cohorts. Clearly, for cohorts born in the first part of the 20th century, the MAB has decreased by about 3 years (about 10 %) in just 25 (annual) cohorts. Moreover, for cohorts born in the second part of the 20th century, a very quick and massive postponement of births, with a 4-year rise in just 25 (annual) cohorts. When transposed to our 3-period OLG model, those changes imply strong variations in MAB over just a few cohorts. An even larger challenge comes from the fact that, for cohorts born in the 20th century, the TFR has remained relatively stable. One must thus find a way to reconcile strong changes in the *tempo* of births with low changes in the *quantum* of births.

Given that, in our model, periods are of about 18 years (which implies having early children at age 18 and late children at age 36), we will focus here on the TFR and MAB for cohorts born in years 1876, 1894, 1912, 1930, 1948 and 1966. Figures 6 and 7 below show the actual levels of TFR and MAB for those cohorts (in continuous traits), as well as the levels of those variables that are simulated under some particular vector of values for the structural parameters of our economy (in discontinuous traits).[9] Note that, unlike in the theoretical model, we have here assumed cohort-specific time cost of children q_t, in such a way as to be able to perfectly fit the cohort TFR pattern (Figure 6).[10] Given that our main focus is on the timing of births rather than on the quantum, and given that the *tempo* of births is invariant to the parameter q, this does not constitute a strong restriction for the purpose at hand.

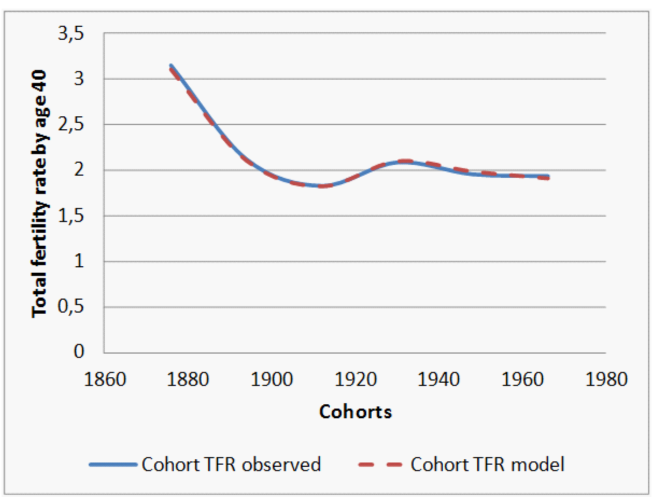

Figure 6: Cohort TFR: data and model.

[9]Figures 6 and 7 rely on the following values for parameters. $h_0 = 0.001$, $v = 3.50$, $\alpha = 0.55$, $\beta = 0.61$, $\gamma = 0.0065$, $\delta = 0.0035$, $\varepsilon = 0.02$, $\eta = 10.55$, $\rho = 0.0075$, $\sigma = 0.66$.

[10]Figure 6 relies on period-specific cost of children: $q_0 = 0.725$, $q_1 = 0.322$, $q_2 = 0.125$, $q_3 = 0.042$, $q_4 = 0.026$, $q_5 = 0.025$.

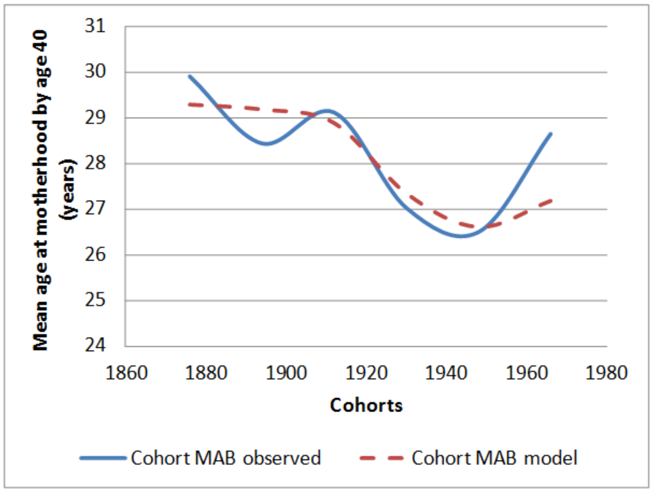

Figure 7: Cohort MAB: data and model

As shown on Figure 7, our model is able to replicate, over the Swedish cohorts considered, the non-monotonic, U-shaped pattern of cohort mean age at birth. Clearly, the model can replicate that, as the economy develops, there is first a process of advancing births, and, then, of postponing births.

The replication of this non-monotonic MAB pattern without using any exogenous shock is achieved by means of the multiplicity of regimes in our model. The economy starts in Regime I (no higher education) for cohorts born in 1876, 1894 and 1912. In that regime, the rise in incomes generates an advancement of births, that is, a fall of the MAB. Then, the economy enters Regime II (positive higher education) for women born in cohort 1930. During that regime, there is also an advancement of births. The economy enters Regime III for cohorts born after 1948, for which the rise in income will be associated not with an advancement, but with a postponement of births.

In the light of this, it appears that our unified growth model can rationalize the dynamics of the *tempo* of births as a succession of regime shifts. Besides those strengths, it is also worth noticing some limitations of our numerical analysis. First, our model can only replicate the overall trend of the MAB, without being able to replicate short-term fluctuations in the MAB, such as the (transitory) postponement of births for cohorts born in the 1900s. However, given that our analysis focuses on long-run trends, this limitation is not problematic. A more important comment on Figure 7 concerns the fact that, although the model can replicate the U-shaped pattern of the MAB, it exhibits nonetheless, from a quantitative perspective, some limitations in being able to reproduce the very strong growth in the MAB just after the reversal. The model captures the non-monotonicity of the MAB pattern, but can hardly reproduce such a strong reversal in a few cohorts. This suggests that other factors - not present in our model - may have also affected the magnitude of the recent births postponement.

6 Conclusions

In the recent decades, economists have paid a substantial attention to the explanation or rationalization of the dynamics of the *quantum* of births, in relation with economic development. Those analyses were most successful in providing explanations for the long-run declining trend in fertility, such as the rise of the opportunity cost of children, a change in the direction of intergenerational transfers, or the fall of infant mortality. However, most analyses were made in models with a unique fertility period. As such, those analyses could bring little light on the dynamics of the *tempo* of births, and on its relationship with economic development.

The goal of this paper was precisely to study the relationship between economic development, the *quantum* of births and the *tempo* of births. Our purpose was to build a dynamic model of lifecycle fertility that can rationalize the observed patterns of the *quantum* and the *tempo* of births. In particular, our goal was to build a model that can explain that, as the economy develops, the mean age at birth tends first to decline, and, then, tends to grow.

For that purpose, we developed a 3-period OLG model with lifecycle fertility (i.e. 2 fertility periods), where individuals choose not only their education and the total number of children, but, also, how they allocate births along their lifecycle. Following the increasingly large unified growth literature, we considered preferences that are sufficiently general so as to allow for multiple types of temporary equilibria, leading to different regimes.

When studying that model, we showed that, depending on the prevailing level of human capital, an economy could be in three distinct regimes. In Regime I, there is no education, and fertility is high. In that traditional regime, a rise in income pushes towards advancing births, since this raises relatively more the opportunity cost of late births in comparison to early births. Then, once human capital reaches some threshold, individuals start investing in education. In that second regime, fertility is lower, but it is still the case that, as income grows, births are being advanced. However, once the human capital stock is sufficiently large so as to reach a second threshold, the relationship between income growth and births postponement is reversed, and higher incomes lead now to postponing births (unlike in Regimes I and II).

The identification of three distinct regimes casts significant light on the relation between economic development and fertility behaviors (*quantum* and *tempo* of births). First of all, our model can explain why advanced industrialized economies exhibit, since the 1970s, both low fertility and births postponement. This coincides with Regime III examined in our model. But our framework can also cast some light on the situation of other, less advanced countries. Some developing countries, for instance in Africa, exhibit declining fertility trends, but still face strong economic difficulties, which prevent the economic take-off. Our model provides an explanation for this situation: those countries remain in Regime II, where births still arise quite early in life, which prevents large investment in education, and, hence, limits the possibility of an economic take-off. Thus, from the perspective of understanding the emergence of an economic take-off, focusing only on the quantity of births does not suffice, since the timing of births matters as much as the quantity of births.

Finally, in order to have a more precise idea of the extent to which our model can fit the data, we also illustrated our model numerically, by focusing on the case of Swedish cohorts born between 1876 and 1966. We showed that our model can approximately replicate the observed U-shaped pattern of the mean age at birth. Our model provides a simple way to rationalize the substantial changes in birth timing that were observed during the 20th century. Although our numerical simulations illustrate the explanatory potential of our model, it remains true that the model can only replicate long-run trends, and not short-run fluctuations. Moreover, the model cannot fully replicate the magnitude of the rise in MAB observed for cohorts born after 1960. This suggests that other factors may have been at work in the postponement of births. Hence, much work remains to be done to develop a more general model of lifecycle fertility.

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8 Appendix

8.1 Proof of Proposition 1

In this Appendix, we show that, when the structural parameters satisfy the conditions:

- $\varepsilon > \delta v$, $\beta\eta > \sigma v$, $\sigma > \beta$, $\Omega < 0$, $\varphi > \left|\bar{h}\Omega\right|$,
- $\frac{\delta}{\varepsilon}\frac{2\Omega^2+4\omega v(\eta\beta-v\sigma)}{4\omega^2} > \left(v+\frac{-\Omega}{2\omega}\right)\left(1-\frac{\delta v}{\varepsilon}\right)+\sqrt[2]{\frac{\Omega^2+4\omega v(\eta\beta-v\sigma)}{4\omega^2}}\left(1-2\frac{\delta}{\varepsilon}\left(v+\frac{-\Omega}{2\omega}\right)\right)$,
- $2\bar{h}\omega\left[v\left(\eta\beta-v\sigma\right)\bar{h}\right] > \varphi\left(\bar{h}\Omega+\varphi\right)$,
- $\frac{\left(\Delta(\tilde{h})\right)^{-1/2}\Delta'(\tilde{h})}{2\tilde{h}}+\delta\frac{\sqrt[2]{\Delta(\tilde{h})}-\varphi}{\left(\tilde{h}\right)^3} < \frac{\left(\Delta(\tilde{h})\right)^{-3/2}\left[\Delta'(\tilde{h})\right]^2}{8}\left(1+\frac{\delta}{\tilde{h}}\right)$,

then the human capital level defines the three regimes described in Proposition 1.

Existence of a threshold for Regime II From the FOC for early fertility, we have:

$$n_t = \frac{\gamma\left[h_t+\delta\right]}{h_t q(\alpha+\gamma)} > 0$$

From the FOC for late fertility, we have:

$$m_{t+1} = \frac{\rho\left[(v+e_t)h_t+\varepsilon\right]}{(v+e_t)h_t q\left[\beta+\rho\right]} > 0$$

From the FOC for education,

$$\frac{\sigma}{(e_t+\eta)} = \frac{\beta h_t(1-m_{t+1}q)}{(v+e_t)h_t(1-m_{t+1}q)+\varepsilon}$$

we obtain:

$$e_t = \frac{\beta h_t(1-m_{t+1}q)\eta-\sigma\varepsilon-\sigma v h_t(1-m_{t+1}q)}{(1-m_{t+1}q)\left(\sigma h_t-\beta h_t\right)}$$

Note that education equals zero when:

$$\beta h_t(1-m_{t+1}q)\eta < \sigma\varepsilon+\sigma v h_t(1-m_{t+1}q)$$

Substituting for m_{t+1} when $e_t = 0$, that expression becomes:

$$\begin{aligned}
\beta h_t\left(1-\frac{\rho\left[vh_t+\varepsilon\right]}{vh_t q\left[\beta+\rho\right]}q\right)\eta &< \sigma\varepsilon+\sigma v h_t\left(1-\frac{\rho\left[vh_t+\varepsilon\right]}{vh_t q\left[\beta+\rho\right]}q\right)\\
h_t\left(\frac{vh_t\left[\beta+\rho\right]-\rho\left[vh_t+\varepsilon\right]}{vh_t\left[\beta+\rho\right]}\right)(\beta\eta-\sigma v) &< \sigma\varepsilon\\
h_t &< \frac{\sigma\varepsilon v\left[\beta+\rho\right]+\rho\varepsilon\left(\beta\eta-\sigma v\right)}{v\beta\left(\beta\eta-\sigma v\right)}\\
h_t &< \frac{\varepsilon\left(\sigma v+\rho\eta\right)}{v\left(\beta\eta-\sigma v\right)} = \bar{h}
\end{aligned}$$

Hence, when $h_t < \bar{h}$, we have $e_t^I = 0$ and $n_t^I = \frac{\gamma[h_t+\delta]}{h_t q(\alpha+\gamma)} > 0$ and $m_{t+1}^I = \frac{\rho[vh_t+\varepsilon]}{vh_t q[\beta+\rho]} > 0$. This situation coincides with Regime I.

Note that, in that regime, we obviously have a corner solution for education, so that $\frac{\partial e_t}{\partial h_t} = 0$. We also have

$$\begin{aligned}\frac{\partial n_t}{\partial h_t} &= \frac{-\gamma\delta}{q(\alpha+\gamma)\left[h_t\right]^2} < 0 \\ \frac{\partial m_{t+1}}{\partial h_t} &= \frac{-\rho\varepsilon}{vq\left[\beta+\rho\right]\left[h_t\right]^2} < 0\end{aligned}$$

Note that the ratio of early births over late births, $R_t = \frac{\frac{\gamma[h_t+\delta]}{h_t q(\alpha+\gamma)}}{\frac{\rho[vh_t+\varepsilon]}{vh_t q[\beta+\rho]}} = \frac{\gamma[h_t+\delta]v[\beta+\rho]}{(\alpha+\gamma)\rho[vh_t+\varepsilon]}$, varies with human capital as follows:

$$\begin{aligned}\frac{\partial R_t}{\partial h_t} &= \frac{\gamma v\left[\beta+\rho\right](\alpha+\gamma)\rho\left[vh_t+\varepsilon\right] - \gamma\left[h_t+\delta\right]v\left[\beta+\rho\right](\alpha+\gamma)\rho v}{\left[(\alpha+\gamma)\rho\left[vh_t+\varepsilon\right]\right]^2} \\ &= v\gamma\left[\beta+\rho\right]\frac{\varepsilon - v\delta}{(\alpha+\gamma)\rho\left[\left[vh_t+\varepsilon\right]\right]^2} > 0\end{aligned}$$

Given the assumption $\varepsilon > v\delta$, we have that, as human capital accumulates, births are being advanced.

Despite the absence of education, human capital grows at a rate v. When $h_t > \bar{h}$, it is no longer the case that education equals zero. Indeed, education equals zero when:

$$\frac{\sigma}{\eta} > \frac{\beta h_t(1 - \frac{\rho[vh_t+\varepsilon]}{vh_t q[\beta+\rho]}q)}{vh_t(1 - \frac{\rho[vh_t+\varepsilon]}{vh_t q[\beta+\rho]}q) + \varepsilon} \iff h_t < \frac{\varepsilon\left(\sigma v + \rho\eta\right)}{v\left(\beta\eta - \sigma v\right)} = \bar{h}$$

Thus when $h_t > \bar{h}$, that strict inequality cannot hold any more. It can be shown that, when $h_t > \bar{h}$, we necessarily have an interior education level $e_t > 0$. To see this, let us first write down the FOC for e_t:

$$\frac{\sigma}{(e_t+\eta)} = \frac{\beta h_t(1 - m_{t+1}q)}{(v+e_t)h_t(1 - m_{t+1}q) + \varepsilon}$$

Substituting for $m_{t+1} = \frac{\rho[(v+e_t)h_t+\varepsilon]}{(v+e_t)h_t q[\beta+\rho]}$ in that FOC, we obtain:

$$\sigma\left[\frac{(v+e_t)h_t\beta - \rho\varepsilon + \varepsilon\left(\beta+\rho\right)}{(\beta+\rho)})\right] = \beta\left(\frac{(v+e_t)h_t\beta - \rho\varepsilon}{(v+e_t)\left(\beta+\rho\right)}\right)(e_t+\eta)$$

Hence

$$\begin{aligned}& e_t^2\left[h_t\beta\sigma - h_t\beta\beta\right] \\ & +e_t\left[2vh_t\beta\sigma - \sigma\rho\varepsilon + \sigma\varepsilon\left(\beta+\rho\right) - (v+\eta)h_t\beta\beta + \beta\rho\varepsilon\right] \\ & -v\eta h_t\beta\beta + \beta\rho\varepsilon\eta + v^2 h_t\beta\sigma - v\sigma\varepsilon\rho + v\sigma\varepsilon\left(\beta+\rho\right) \\ =\ & 0\end{aligned}$$

Hence

$$\begin{aligned} & e_t^2 h_t [\sigma - \beta] + e_t [h_t (2v\sigma - (v+\eta)\beta) + \varepsilon (\sigma + \rho)] \\ & -vh_t (\eta\beta - v\sigma) + \varepsilon (\rho\eta + v\sigma) \\ = \; & 0 \end{aligned}$$

Hence we have

$$\begin{aligned} \Delta(h_t) \;=\; & [h_t (2v\sigma - (v+\eta)\beta) + \varepsilon (\sigma + \rho)]^2 \\ & -4h_t [\sigma - \beta] [-vh_t (\eta\beta - v\sigma) + \varepsilon (\rho\eta + v\sigma)] \end{aligned}$$

The first term is positive. Given that the threshold is $\frac{\varepsilon(\sigma v+\rho\eta)}{v(\beta\eta-\sigma v)} > h_t$, we have here $\frac{\varepsilon(\sigma v+\rho\eta)}{v(\beta\eta-\sigma v)} < h_t$, which implies that $[-vh_t (\eta\beta - v\sigma) + \varepsilon (\rho\eta + v\sigma)] < 0$. Hence, provided $\sigma - \beta > 0$, we have $\Delta(h_t) > 0$.

Note that, given $\bar{h} = \frac{\varepsilon(\sigma v+\rho\eta)}{v(\beta\eta-\sigma v)}$, we can rewrite the above expression as:

$$\Delta (h_t) = [h_t (2v\sigma - (v+\eta)\beta) + \varepsilon (\sigma + \rho)]^2 + 4h_t [\sigma - \beta] \left[v (\eta\beta - v\sigma) \left(h_t - \bar{h}\right)\right]$$

Using the notations $\Omega \equiv 2v\sigma - (v+\eta)\beta$, $\varphi \equiv \varepsilon (\sigma + \rho)$ and $\omega \equiv (\sigma - \beta)$, we have:

$$\Delta (h_t) = [h_t\Omega + \varphi]^2 + 4h_t\omega \left[v (\eta\beta - v\sigma) \left(h_t - \bar{h}\right)\right]$$

with

$$\begin{aligned} \Delta' (h_t) &= 2 [h_t\Omega + \varphi] \Omega + 4\omega \left[v (\eta\beta - v\sigma) \left(h_t - \bar{h}\right)\right] + 4h_t\omega [v (\eta\beta - v\sigma)] \\ &= 2 [h_t\Omega + \varphi] \Omega + 8h_t\omega [v (\eta\beta - v\sigma)] - 4\omega \left[v (\eta\beta - v\sigma) \bar{h}\right] \end{aligned}$$

We can rewrite education as:

$$e_t^{II} = \frac{-[h_t\Omega + \varphi] + \sqrt[2]{\Delta (h_t)}}{2h_t\omega}$$

Notice that, when $h_t > \bar{h}$, we have $\Delta (h_t) > 0$ and $\sqrt[2]{\Delta (h_t)} > [h_t\Omega + \varphi]$, so that education is necessarily positive. Hence we have $e_t^{II} > 0 = e_t^I$.

We have also, in that second regime,

$$\begin{aligned} n_t^{II} &= \frac{\gamma [h_t + \delta]}{h_t q(\alpha + \gamma)} \\ m_{t+1}^{II} &= \frac{\rho \left[(v + \frac{-[h_t\Omega+\varphi]+\sqrt[2]{\Delta(h_t)}}{2h_t\omega})h_t + \varepsilon\right]}{(v + \frac{-[h_t\Omega+\varphi]+\sqrt[2]{\Delta(h_t)}}{2h_t\omega})h_t q [\beta + \rho]} \\ &= \frac{\rho \left[2vh_t\omega - [h_t\Omega + \varphi] + \sqrt[2]{\Delta (h_t)} + 2\omega\varepsilon\right]}{q [\beta + \rho] (2h_t\omega v - [h_t\Omega + \varphi] + \sqrt[2]{\Delta (h_t)})} \end{aligned}$$

Given that h_t is higher in Regime II than in Regime I, we obviously have $n_t^{II} < n_t^I$.

Regarding m_{t+1}^{II}, we have:

$$\begin{aligned}\frac{\partial m_{t+1}^{II}}{\partial h_t} &= \frac{\begin{array}{c}\rho\left[2v\omega-\Omega+\frac{1}{2}\left[\Delta\left(h_t\right)\right]^{-1/2}\Delta'\left(h_t\right)\right]q\left[\beta+\rho\right]\left(2h_t\omega v-\left[h_t\Omega+\varphi\right]+\sqrt[2]{\Delta\left(h_t\right)}\right)\\ -\rho\left[2vh_t\omega-\left[h_t\Omega+\varphi\right]+\sqrt[2]{\Delta\left(h_t\right)}+2\omega\varepsilon\right]q\left[\beta+\rho\right]\left(2\omega v-\Omega+\frac{1}{2}\left[\Delta\left(h_t\right)\right]^{-1/2}\Delta'\left(h_t\right)\right)\end{array}}{\left[q\left[\beta+\rho\right]\left(2h_t\omega v-\left[h_t\Omega+\varphi\right]+\sqrt[2]{\Delta\left(h_t\right)}\right)\right]^2} \\ &= \rho\frac{-\left[2\omega\varepsilon\right]\left(2\omega v-\Omega+\frac{1}{2}\left[\Delta\left(h_t\right)\right]^{-1/2}\Delta'\left(h_t\right)\right)}{q\left[\beta+\rho\right]\left[\left(2h_t\omega v-\left[h_t\Omega+\varphi\right]+\sqrt[2]{\Delta\left(h_t\right)}\right)\right]^2}<0\end{aligned}$$

Regarding the impact of human capital on the ratio R_t, we now have:

$$R_t=\frac{\frac{\gamma[h_t+\delta]}{h_t q(\alpha+\gamma)}}{\frac{\rho[(v+e(h))h_t+\varepsilon]}{(v+e(h))h_t q[\beta+\rho]}}=\frac{\gamma\left[h_t+\delta\right]\left(v+e(h_t)\right)\left[\beta+\rho\right]}{(\alpha+\gamma)\rho\left[(v+e(h_t))h_t+\varepsilon\right]}$$

where $e\left(h_t\right)\equiv\frac{-[h_t\Omega+\varphi]+\sqrt[2]{\Delta(h_t)}}{2h_t\omega}$.

We have:

$$\begin{aligned}\frac{\partial R_t}{\partial h_t} &= \frac{\begin{array}{c}\left[\gamma(v+e(h_t))\left(\beta+\rho\right)+\gamma\left(h_t+\delta\right)\left(e'(h_t)\right)\left(\beta+\rho\right)\right]\left(\alpha+\gamma\right)\rho((v+e(h_t))h_t+\varepsilon)\\ -\left[\gamma\left(h_t+\delta\right)\left(v+e(h_t)\right)\left(\beta+\rho\right)\right]\left[\left(\alpha+\gamma\right)\rho(e'(h_t)h_t+v+e(h_t))\right]\end{array}}{\left[\left(\alpha+\gamma\right)\rho((v+e(h_t))h_t+\varepsilon)\right]^2} \\ &= \left(\beta+\rho\right)\gamma\left(\alpha+\gamma\right)\frac{\begin{array}{c}\left[(v+e(h_t))+\left(h_t+\delta\right)\left(e'(h_t)\right)\right]\rho((v+e(h_t))h_t+\varepsilon)\\ -\left[\left(h_t+\delta\right)\left(v+e(h_t)\right)\right]\left[\rho(e'(h_t)h_t+v+e(h_t))\right]\end{array}}{\left[\left(\alpha+\gamma\right)\rho((v+e(h_t))h_t+\varepsilon)\right]^2} \\ &= \left(\beta+\rho\right)\gamma\frac{\varepsilon\left[v+e(h_t)+\left(h_t+\delta\right)e'(h_t)\right]-\delta(v+e(h_t))^2}{\left(\alpha+\gamma\right)\rho\left[(v+e(h_t))h_t+\varepsilon\right]^2}\end{aligned}$$

We have thus:

$$\frac{\partial R_t}{\partial h_t}>0\iff\left(\beta+\rho\right)\gamma\frac{\varepsilon\left[v+e(h_t)+\left(h_t+\delta\right)e'(h_t)\right]-\delta(v+e(h_t))^2}{\left(\alpha+\gamma\right)\rho\left[(v+e(h_t))h_t+\varepsilon\right]^2}>0$$

This implies:

$$\begin{aligned}\varepsilon\left[v+e(h_t)+\left(h_t+\delta\right)e'(h_t)\right]-\delta(v+e(h_t))^2 &> 0\\ \varepsilon\left[v+e(h_t)+\left(h_t+\delta\right)e'(h_t)\right] &> \delta(v+e(h_t))^2\\ \varepsilon\left[v+e(h_t)+\left(h_t+\delta\right)e'(h_t)\right] &> \delta(v+e(h_t))^2\end{aligned}$$

When education equals 0 (Regime I), that corner solution is insensitive to h_t, and we have thus that the condition vanishes to $\varepsilon>\delta v$, which is necessarily satisfied given our assumption on structural parameters. Thus, when $h_t<\bar{h}$, it is necessarily the case that $\frac{\partial R_t}{\partial h_t}>0$.

Regarding the impact of human capital on education, note that:

$$e(h_t) = \frac{-\left[h_t\Omega + \varphi\right] + \sqrt[2]{\Delta(h_t)}}{2h_t\omega}$$

We have thus:

$$\begin{aligned}
e'(h_t) &= \frac{\left[-\Omega + \frac{1}{2}\left[\Delta(h_t)\right]^{-1/2}\Delta'(h_t)\right]h_t - \left[-\left[h_t\Omega+\varphi\right] + \sqrt[2]{\Delta(h_t)}\right]}{2\omega\left(h_t\right)^2} \\
&= \frac{\frac{1}{2}\left[\Delta(h_t)\right]^{-1/2}\Delta'(h_t)h_t - \sqrt[2]{\Delta(h_t)} + \varphi}{2\omega\left(h_t\right)^2} \\
&= \frac{\frac{\sqrt[2]{\Delta(h_t)}}{2}\frac{\Delta'(h_t)h_t}{\Delta(h_t)} - \sqrt[2]{\Delta(h_t)} + \varphi}{2\omega\left(h_t\right)^2} \\
&= \frac{\sqrt[2]{\Delta(h_t)}\left(\frac{1}{2}\frac{\Delta'(h_t)h_t}{\Delta(h_t)} - 1\right) + \varphi}{2\omega\left(h_t\right)^2}
\end{aligned}$$

It is easy to show that $\frac{\Delta'(h_t)h_t}{\Delta(h_t)} > 2$, so that $e'(h_t) > 0$. Note that,

$$\begin{aligned}
\frac{\Delta'(h_t)h_t}{\Delta(h_t)} &= h_t\frac{2\left[h_t\Omega+\varphi\right]\Omega + 4\omega\left[v\left(\eta\beta - v\sigma\right)\left(h_t - \bar{h}\right)\right] + 4h_t\omega\left[v\left(\eta\beta - v\sigma\right)\right]}{\left[h_t\Omega+\varphi\right]^2 + 4h_t\omega\left[v\left(\eta\beta - v\sigma\right)\left(h_t - \bar{h}\right)\right]} \\
&= \frac{2\left[h_t^2\Omega^2 + \varphi h_t\Omega\right] + 4h_t\omega\left[v\left(\eta\beta - v\sigma\right)\left(h_t - \bar{h}\right)\right] + 4h_t^2\omega\left[v\left(\eta\beta - v\sigma\right)\right]}{\left[h_t\Omega+\varphi\right]^2 + 4h_t\omega\left[v\left(\eta\beta - v\sigma\right)\left(h_t - \bar{h}\right)\right]}
\end{aligned}$$

Hence, $\frac{\Delta'(h_t)h_t}{\Delta(h_t)} > 2$ implies:

$$\begin{aligned}
& 2\left[h_t^2\Omega^2 + \varphi h_t\Omega\right] + 4h_t\omega\left[v\left(\eta\beta - v\sigma\right)\left(h_t - \bar{h}\right)\right] + 4h_t^2\omega\left[v\left(\eta\beta - v\sigma\right)\right] \\
> \; & 2\left[h_t\Omega+\varphi\right]^2 + 8h_t\omega\left[v\left(\eta\beta - v\sigma\right)\left(h_t - \bar{h}\right)\right]
\end{aligned}$$

Or, alternatively:

$$0 > 2\varphi\left(h_t\Omega + \varphi\right) - 4h_t\omega\left[v\left(\eta\beta - v\sigma\right)\bar{h}\right]$$

Note that the RHS is decreasing in h_t, since $\Omega < 0$. Hence a sufficient condition for this inequality to hold for any $h_t \geq \bar{h}$ is thus:

$$0 > 2\varphi\left(\bar{h}\Omega + \varphi\right) - 4\bar{h}\omega\left[v\left(\eta\beta - v\sigma\right)\bar{h}\right]$$

Under this condition, education is growing in human capital for all $h_t \geq \bar{h}$.

Existence of a threshold for Regime III In the remaining of this proof, we show that there exists one level of human capital $\tilde{h} > \bar{h}$ that is such that, at $\tilde{h}$, we have $\frac{\partial R_t}{\partial h_t} = 0$ and that, for any $h_t > \tilde{h}$, we have $\frac{\partial R_t}{\partial h_t} < 0$. This threshold

$\tilde{h}$ defines the entrance of the economy in Regime III, where the relation between birth timing and human capital accumulation is reversed.

In order to prove the existence and uniqueness of that threshold $\tilde{h} > \bar{h}$, let us first rewrite the condition for $\frac{\partial R_t}{\partial h_t} > 0$ as:

$$e(h_t) + (h_t + \delta)\, e'(h_t) > \frac{\delta}{\varepsilon}(v + e(h_t))^2 - v$$

When h_t tends to 0, the LHS tends to 0 and RHS tends to $v\left(\frac{\delta}{\varepsilon}v - 1\right) < 0$ since $\varepsilon > \delta v$. Thus the condition is trivially satisfied, and thus $\frac{\partial R_t}{\partial h_t} > 0$.

When h_t tends to $\bar{h}$, we have:

$$LHS = 0 + \left(\bar{h} + \delta\right) e'(\bar{h})$$

Note that

$$e(h_t) = \frac{-\left[h_t \Omega + \varphi\right] + \sqrt[2]{\Delta(h_t)}}{2 h_t \omega}$$

Hence, at $h_t = \bar{h}$, we have:

$$\begin{aligned} e'\left(\bar{h}\right) &= \frac{\varphi}{2\omega\left(\bar{h}\right)^2} + \frac{\frac{1}{2}\left[\Delta(\bar{h})\right]^{-1/2}\Delta'(\bar{h})\bar{h} - \sqrt[2]{\Delta(\bar{h})}}{2\omega\left(\bar{h}\right)^2} \\ &= \frac{\varphi}{2\omega\left(\bar{h}\right)^2} + \frac{\frac{1}{2}\left[\left[\bar{h}\Omega + \varphi\right]^2\right]^{-1/2}\Delta'(\bar{h})\bar{h} - \sqrt[2]{\left[\bar{h}\Omega + \varphi\right]^2}}{2\omega\left(\bar{h}\right)^2} \\ &= \frac{\varphi}{2\omega\left(\bar{h}\right)^2} + \frac{\Delta'(\bar{h})\bar{h} - 2\left[\bar{h}\Omega + \varphi\right]^2}{4\omega\left(\bar{h}\right)^2\left[\bar{h}\Omega + \varphi\right]} \end{aligned}$$

We have $\Delta'(\bar{h}) = 2\left[h_t\Omega + \varphi\right]\Omega + 4\omega v\left(\eta\beta - v\sigma\right)\left[2h_t - \bar{h}\right] = 2\left[\bar{h}\Omega + \varphi\right]\Omega + 4\omega v\left(\eta\beta - v\sigma\right)\bar{h}$

Hence

$$\begin{aligned} e'\left(\bar{h}\right) &= \frac{\varphi}{2\omega\left(\bar{h}\right)^2} + \frac{\left[2\left[\bar{h}\Omega + \varphi\right]\Omega + 4\omega v\left(\eta\beta - v\sigma\right)\bar{h}\right]\bar{h} - 2\left[\bar{h}\Omega + \varphi\right]^2}{4\omega\left(\bar{h}\right)^2\left[\bar{h}\Omega + \varphi\right]} \\ &= \frac{\varphi}{2\omega\left(\bar{h}\right)^2} + \frac{\left[\left[\bar{h}\bar{h}\Omega + \varphi\bar{h}\right]\Omega + 2\omega v\left(\eta\beta - v\sigma\right)\bar{h}\bar{h}\right] - \left[\left(\bar{h}\Omega\right)^2 + 2\bar{h}\Omega\varphi + \varphi^2\right]}{2\omega\left(\bar{h}\right)^2\left[\bar{h}\Omega + \varphi\right]} \\ &= \frac{\varphi}{2\omega\left(\bar{h}\right)^2} + \frac{-\varphi\bar{h}\Omega + 2\omega v\left(\eta\beta - v\sigma\right)\bar{h}\bar{h} - \varphi^2}{2\omega\left(\bar{h}\right)^2\left[\bar{h}\Omega + \varphi\right]} \\ &= \frac{v\left(\eta\beta - v\sigma\right)}{\left[\bar{h}\Omega + \varphi\right]} > 0 \end{aligned}$$

which is positive, since $\varphi > \left|\bar{h}\Omega\right|$.

Hence the LHS is:

$$LHS = 0 + \left(\bar{h} + \delta\right) \frac{v\left(\eta\beta - v\sigma\right)}{\left[\bar{h}\Omega + \varphi\right]} > 0$$

The RHS is, at $h_t = \bar{h}$:

$$\begin{aligned} RHS &= \frac{\delta}{\varepsilon}(v + e(\bar{h}))^2 - v \\ &= \frac{\delta}{\varepsilon}(v + 0)^2 - v \\ &= v\left(\frac{\delta}{\varepsilon}v - 1\right) < 0 \end{aligned}$$

Thus, at $h = \bar{h}$, it is also the case that the LHS exceeds the RHS, implying $\frac{\partial R_t}{\partial h_t} > 0$.

Let us now examine how the LHS and the RHS of the expression:

$$e(h_t) + (h_t + \delta)\, e'(h_t) = \frac{\delta}{\varepsilon}(v + e(h_t))^2 - v$$

vary when $h_t \to +\infty$. Clearly, given that the LHS exceeds the RHS for $h \leq \bar{h}$, a sufficient condition for the existence of a threshold $\tilde{h}$ at which $\frac{\partial R_t}{\partial h_t} = 0$ (that is, at which the LHS and the RHS are equal) is that the RHS exceeds the LHS when $h_t \to +\infty$. Then, by continuity, we would be able to conclude that the LHS and RHS intersect for some value of h_t, which is precisely the threshold $\tilde{h}$

We have:

$$\lim_{h_t \to \infty} LHS = \lim_{h_t \to \infty} \left[e(h_t) + (h_t + \delta)\, e'(h_t)\right]$$

Let us decompose this in two components:

$$\lim_{h_t \to \infty} LHS = \lim_{h_t \to \infty} e(h_t) + \lim_{h_t \to \infty} (h_t + \delta)\, e'(h_t)$$

Regarding the first component, we have, using the Hospital Rule:

$$
\begin{aligned}
\lim_{h_t\to\infty} e(h_t) &= \lim_{h_t\to\infty} \frac{-\left[h_t\Omega+\varphi\right]+\sqrt[2]{\Delta(h_t)}}{2h_t\omega} \\
&= \frac{-\Omega}{2\omega} - 0 + \lim_{h_t\to\infty} \frac{\sqrt[2]{\Delta(h_t)}}{2h_t\omega} \\
&= \frac{-\Omega}{2\omega} - 0 + \lim_{h_t\to\infty} \frac{\sqrt[2]{\left[h_t\Omega+\varphi\right]^2 + 4h_t\omega\left[v\left(\eta\beta - v\sigma\right)\left(h_t-\bar{h}\right)\right]}}{2h_t\omega} \\
&= \frac{-\Omega}{2\omega} - 0 + \lim_{h_t\to\infty} \sqrt[2]{\frac{\left[h_t\Omega+\varphi\right]^2 + 4h_t\omega\left[v\left(\eta\beta - v\sigma\right)\left(h_t-\bar{h}\right)\right]}{4h_t^2\omega^2}} \\
&= \frac{-\Omega}{2\omega} - 0 + \sqrt[2]{\lim_{h_t\to\infty} \frac{\left[h_t\Omega+\varphi\right]^2 + 4h_t\omega\left[v\left(\eta\beta - v\sigma\right)\left(h_t-\bar{h}\right)\right]}{4h_t^2\omega^2}} \\
&= \frac{-\Omega}{2\omega} - 0 + \sqrt[2]{\lim_{h_t\to\infty} \frac{2\left[h_t\Omega+\varphi\right]\Omega + 4\omega\left[v\left(\eta\beta - v\sigma\right)\left(2h_t-\bar{h}\right)\right]}{8h_t\omega^2}} \\
&= \frac{-\Omega}{2\omega} - 0 + \sqrt[2]{\lim_{h_t\to\infty} \frac{2\Omega\Omega + 4\omega\left[v\left(\eta\beta - v\sigma\right)2\right]}{8\omega^2}} \\
&= \frac{-\Omega}{2\omega} + \sqrt[2]{\frac{2\Omega^2 + 8\omega v\left(\eta\beta - v\sigma\right)}{8\omega^2}} > 0
\end{aligned}
$$

Regarding the second component, it is straightforward to show that it is equal to:

$$\lim_{h_t\to\infty} \left(h_t+\delta\right) e'(h_t) = 0$$

since

$$\lim_{h_t\to\infty} e'(h_t) = 0$$

Indeed, since $e(h_t) = \frac{-\left[h_t\Omega+\varphi\right]+\sqrt[2]{\Delta(h_t)}}{2h_t\omega}$, we have:

$$
\begin{aligned}
e'(h_t) &= \frac{\varphi}{2\omega h_t^2} + \frac{\frac{1}{2}\left[\Delta(h_t)\right]^{-1/2}\Delta'(h_t)h_t - \sqrt[2]{\Delta(h_t)}}{2h_t^2\omega} \\
e'(h_t) &= \frac{1}{2\omega h_t^2}\left[\varphi + \frac{1}{2}\sqrt[2]{\Delta(h_t)}\left(\frac{\Delta'(h_t)h_t}{\Delta(h_t)} - 1\right)\right]
\end{aligned}
$$

Hence

$$\lim_{h_t\to\infty} e'(h_t) = 0\left[\varphi + \lim_{h_t\to\infty} \frac{1}{2}\sqrt[2]{\Delta(h_t)}\left(\frac{\Delta'(h_t)h_t}{\Delta(h_t)} - 1\right)\right] = 0$$

We thus have:

$$\lim_{h_t\to\infty} LHS = \frac{-\Omega}{2\omega} + \sqrt[2]{\frac{2\Omega^2 + 8\omega v\left(\eta\beta - v\sigma\right)}{8\omega^2}}$$

Concerning the RHS, we have:

$$\lim_{h_t \to \infty} RHS = \lim_{h \to \infty} \frac{\delta}{\varepsilon}(v + e(h_t))^2 - v$$

Given that:

$$\lim_{h_t \to \infty} e(h_t) = \frac{-\Omega}{2\omega} + \sqrt[2]{\frac{2\Omega^2 + 8\omega v\left(\eta\beta - v\sigma\right)}{8\omega^2}}$$

we have:

$$\lim_{h_t \to \infty} RHS = \frac{\delta}{\varepsilon}\left(v + \frac{-\Omega}{2\omega} + \sqrt[2]{\frac{2\Omega^2 + 8\omega v\left(\eta\beta - v\sigma\right)}{8\omega^2}}\right)^2 - v$$

Hence a sufficient condition for the RHS to be above the LHS when $h_t \to \infty$ is:

$$\begin{array}{rcl}
\frac{\delta}{\varepsilon}\left(v + \frac{-\Omega}{2\omega} + \sqrt[2]{\frac{2\Omega^2 + 8\omega v\left(\eta\beta - v\sigma\right)}{8\omega^2}}\right)^2 - v & > & \frac{-\Omega}{2\omega} + \sqrt[2]{\frac{2\Omega^2 + 8\omega v\left(\eta\beta - v\sigma\right)}{8\omega^2}} \\
\frac{\delta}{\varepsilon}\left(v + \frac{-\Omega}{2\omega}\right)^2 + \frac{\delta}{\varepsilon}\frac{2\Omega^2 + 8\omega v\left(\eta\beta - v\sigma\right)}{8\omega^2} - v & > & \frac{-\Omega}{2\omega} + \sqrt[2]{\frac{2\Omega^2 + 8\omega v\left(\eta\beta - v\sigma\right)}{8\omega^2}}\left(1 - 2\frac{\delta}{\varepsilon}\left(v + \frac{-\Omega}{2\omega}\right)\right) \\
\frac{\delta}{\varepsilon}\left(\frac{-\Omega}{2\omega}\right)^2 + \frac{\delta}{\varepsilon}\frac{\Omega^2 + 4\omega v\left(\eta\beta - v\sigma\right)}{4\omega^2} & > & \left[\begin{array}{c} \left(v + \frac{-\Omega}{2\omega}\right)\left(1 - \frac{\delta v}{\varepsilon}\right) \\ + \sqrt[2]{\frac{\Omega^2 + 4\omega v(\eta\beta - v\sigma)}{4\omega^2}}\left(1 - 2\frac{\delta}{\varepsilon}\left(v + \frac{-\Omega}{2\omega}\right)\right) \end{array}\right] \\
\frac{\delta}{\varepsilon}\frac{2\Omega^2 + 4\omega v\left(\eta\beta - v\sigma\right)}{4\omega^2} & > & \left[\begin{array}{c} \left(v + \frac{-\Omega}{2\omega}\right)\left(1 - \frac{\delta v}{\varepsilon}\right) \\ + \sqrt[2]{\frac{\Omega^2 + 4\omega v(\eta\beta - v\sigma)}{4\omega^2}}\left(1 - 2\frac{\delta}{\varepsilon}\left(v + \frac{-\Omega}{2\omega}\right)\right) \end{array}\right]
\end{array}$$

Under that condition on structural parameters, the RHS exceeds the LHS when $h_t \to \infty$. Hence, given that the LHS exceeds the RHS when $h \to \bar{h}$, there must exist, by continuity, at least one intersection between the RHS and the LHS, that is, a threshold $\tilde{h}$ such that:

$$e(\tilde{h}) + \left(\tilde{h} + \delta\right)e'(\tilde{h}) = \frac{\delta}{\varepsilon}(v + e(\tilde{h}))^2 - v$$

Thus, once we make this assumption

$$\frac{\delta}{\varepsilon}\frac{2\Omega^2 + 4\omega v\left(\eta\beta - v\sigma\right)}{4\omega^2} > \left[\begin{array}{c} \left(v + \frac{-\Omega}{2\omega}\right)\left(1 - \frac{\delta v}{\varepsilon}\right) \\ + \sqrt[2]{\frac{\Omega^2 + 4\omega v(\eta\beta - v\sigma)}{4\omega^2}}\left(1 - 2\frac{\delta}{\varepsilon}\left(v + \frac{-\Omega}{2\omega}\right)\right) \end{array}\right]$$

we know for sure that there exists a threshold $\tilde{h}$ such that, for that human capital level, $\frac{dR_t}{dh_t} = 0$.

Uniqueness of the threshold for Regime III In order to be sure that there exists *only one* such threshold $\tilde{h}$, we need to examine the monotonicity of the LHS and RHS of the condition for $\frac{dR_t}{dh_t} > 0$. If LHS strictly decreasing in h_t, and RHS strictly increasing in h_t, then the threshold is unique and we know for sure that for $h_t < \tilde{h}$, $\frac{dR_t}{dh_t} > 0$ and for $h_t > \tilde{h}$, $\frac{dR_t}{dh_t} < 0$.

Regarding the **RHS**, the mononicity condition is:

$$\frac{d\left[\frac{\delta}{\varepsilon}(v + e(h_t))^2 - v\right]}{dh_t} > 0$$

Developing yields:

$$\begin{aligned}
\frac{\delta}{\varepsilon}2(v + e(h_t))e'(h_t) &> 0 \\
\frac{2\delta}{\varepsilon}\left[v + \frac{-\Omega}{2\omega} - \frac{\varphi}{2h_t\omega} + \frac{\sqrt[2]{\Delta(h_t)}}{2h_t\omega}\right]\left[\frac{\varphi}{2\omega\left(h_t\right)^2} + \frac{1}{2}\frac{\left(\Delta(h_t)\right)^{-1/2}\Delta'(h_t)}{2\omega h_t} - \frac{\sqrt[2]{\Delta(h_t)}}{2\omega\left(h_t\right)^2}\right] &> 0 \\
\frac{2\delta}{\varepsilon}\left[v + \frac{-\Omega}{2\omega} + \frac{\sqrt[2]{\Delta(h_t)} - \varphi}{2h_t\omega}\right]\left[\frac{\varphi - \sqrt[2]{\Delta(h_t)}}{2\omega\left(h_t\right)^2} + \frac{\left(\Delta(h_t)\right)^{-1/2}\Delta'(h_t)}{4\omega h_t}\right] &> 0
\end{aligned}$$

The inequality holds when

$$\left[v + \frac{\sqrt[2]{\Delta(h_t)} - (\varphi + \Omega h_t)}{2h_t\omega}\right]\left[\frac{\varphi - \sqrt[2]{\Delta(h_t)}}{2\omega\left(h_t\right)^2} + \frac{\left(\Delta(h_t)\right)^{-1/2}\Delta'(h_t)}{4\omega h_t}\right] > 0$$

Note that, since $\Delta \equiv \left[h_t\Omega + \varphi\right]^2 + 4h_t\omega v\left(\eta\beta - v\sigma\right)\left(h_t - \bar{h}\right)$, we have: $\Delta - 4h_t\omega v\left(\eta\beta - v\sigma\right)\left(h_t - \bar{h}\right) = \left[h_t\Omega + \varphi\right]^2$

Hence $\sqrt[2]{\Delta - 4h_t\omega v\left(\eta\beta - v\sigma\right)\left(h_t - \bar{h}\right)} - h_t\Omega = \varphi$. Thus $\varphi < \sqrt[2]{\Delta(h_t)}$

We have thus $\sqrt[2]{\Delta - 4h_t\omega v\left(\eta\beta - v\sigma\right)\left(h_t - \bar{h}\right)} = \varphi + \Omega h_t < \sqrt[2]{\Delta(h_t)}$ Thus factor 1 of the product is strictly positive.

But first term of Factor 2 is negative.

Thus strict monotonicity is achieved when (sufficient condition):

$$\begin{aligned}
\frac{(\Delta(h_t))^{-1/2}\,\Delta'(h_t)}{4\omega h_t} &> \frac{\sqrt[2]{\Delta(h_t)}-\varphi}{2\omega\,(h_t)^2}\\
\frac{\Delta'(h_t)}{2\sqrt[2]{\Delta(h_t)}} &> \frac{\sqrt[2]{\Delta(h_t)}-\varphi}{h_t}\\
\Delta'(h_t) &> 2\sqrt[2]{\Delta(h_t)}\frac{\sqrt[2]{\Delta(h_t)}-\varphi}{h_t}\\
2\left[h_t\Omega+\varphi\right]\Omega+4\omega v\left(\eta\beta-v\sigma\right)\left(2h_t-\bar{h}\right) &> 2\frac{\left[\begin{array}{c}\left[h_t\Omega+\varphi\right]^2+4h_t\omega v\left(\eta\beta-v\sigma\right)\left(h_t-\bar{h}\right)\\ -\varphi\sqrt[2]{\left[h_t\Omega+\varphi\right]^2+4h_t\omega v\left(\eta\beta-v\sigma\right)\left(h_t-\bar{h}\right)}\end{array}\right]}{h_t}\\
&\quad h_t^2\Omega^2+\varphi h_t\Omega+2\omega v\left(\eta\beta-v\sigma\right)\left(2h_t-\bar{h}\right)h_t\\
&> \left[\begin{array}{c}\left[h_t\Omega+\varphi\right]^2+4h_t\omega v\left(\eta\beta-v\sigma\right)\left(h_t-\bar{h}\right)\\ -\varphi\sqrt[2]{\left[h_t\Omega+\varphi\right]^2+4h_t\omega v\left(\eta\beta-v\sigma\right)\left(h_t-\bar{h}\right)}\end{array}\right]
\end{aligned}$$

Hence the condition is:

$$\begin{aligned}
&2h_t\omega v\left(\eta\beta-v\sigma\right)h_t\\
> \;&\varphi^2+h_t\Omega\varphi+2h_t\omega v\left(\eta\beta-v\sigma\right)\left(h_t-\bar{h}\right)-\varphi\sqrt[2]{\left[h_t\Omega+\varphi\right]^2+4h_t\omega v\left(\eta\beta-v\sigma\right)\left(h_t-\bar{h}\right)}
\end{aligned}$$

Hence

$$\begin{aligned}
0 &> \varphi^2+h_t\Omega\varphi-2h_t\omega v\left(\eta\beta-v\sigma\right)\bar{h}-\varphi\sqrt[2]{\left[h_t\Omega+\varphi\right]^2+4h_t\omega v\left(\eta\beta-v\sigma\right)\left(h_t-\bar{h}\right)}\\
0 &> \varphi(\varphi+h_t\Omega)-2h_t\omega v\left(\eta\beta-v\sigma\right)\bar{h}-\varphi\sqrt[2]{\left[h_t\Omega+\varphi\right]^2+4h_t\omega v\left(\eta\beta-v\sigma\right)\left(h_t-\bar{h}\right)}\\
0 &> \varphi(\sqrt[2]{\left[h_t\Omega+\varphi\right]^2+4h_t\omega v\left(\eta\beta-v\sigma\right)\left(h_t-\bar{h}\right)-4h_t\omega v\left(\eta\beta-v\sigma\right)\left(h_t-\bar{h}\right)})-2h_t\omega v\left(\eta\beta-v\sigma\right)\bar{h}\\
&\quad -\varphi\sqrt[2]{\left[h_t\Omega+\varphi\right]^2+4h_t\omega v\left(\eta\beta-v\sigma\right)\left(h_t-\bar{h}\right)}
\end{aligned}$$

Given that $\varphi(\sqrt[2]{\left[h_t\Omega+\varphi\right]^2+4h_t\omega v\left(\eta\beta-v\sigma\right)\left(h_t-\bar{h}\right)-4h_t\omega v\left(\eta\beta-v\sigma\right)\left(h_t-\bar{h}\right)})<$ $\varphi\sqrt[2]{\left[h_t\Omega+\varphi\right]^2+4h_t\omega v\left(\eta\beta-v\sigma\right)\left(h_t-\bar{h}\right)}$, the RHS is unambiguously negative, and thus the strict monotonicity condition is satisfied for the RHS, which is strictly increasing.

Consider now the **LHS**:

A necessary and sufficient condition for the uniqueness of the threshold is that, at *any* $\hat{h}$ satisfying:

$$e(\hat{h})+\left(\hat{h}+\delta\right)e'(\hat{h})=\frac{\delta}{\varepsilon}(v+e(\hat{h}))^2-v$$

we have that the LHS is strictly decreasing in h:

$$2e'(\hat{h})+\left(\hat{h}+\delta\right)e''(\hat{h})<0$$

Indeed, if it is not the case that the LHS expression is strictly decreasing at any intersection with the RHS expression, then it means that there must be more than one intersection.

Substituting for $e'(\hat{h})$ and $e''(\hat{h})$, the condition becomes:

$$\begin{aligned} & 2\left[\frac{\varphi}{2\omega\left(\hat{h}\right)^{2}}+\frac{1}{2}\frac{\left(\Delta(\hat{h})\right)^{-1/2}\Delta'(\hat{h})}{2\omega\hat{h}}-\frac{\sqrt[2]{\Delta(\hat{h})}}{2\omega\left(\hat{h}\right)^{2}}\right] \\ & +\left(\hat{h}+\delta\right)\left[\frac{-\varphi+\sqrt[2]{\Delta(\hat{h})}}{\omega\left(\hat{h}\right)^{3}}-\frac{\left(\Delta(\hat{h})\right)^{-3/2}\left[\Delta'(\hat{h})\right]^{2}}{8\omega\hat{h}}\right] \\ < \;& 0 \end{aligned}$$

Let us simplify this as:

$$\begin{aligned} & \frac{\left(\Delta(\hat{h})\right)^{-1/2}\Delta'(\hat{h})}{2\omega\hat{h}}-\frac{\left(\Delta(\hat{h})\right)^{-3/2}\left[\Delta'(\hat{h})\right]^{2}}{8\omega} \\ & +\delta\frac{-\varphi+\sqrt[2]{\Delta(\hat{h})}}{\omega\left(\hat{h}\right)^{3}}-\delta\frac{\left(\Delta(\hat{h})\right)^{-3/2}\left[\Delta'(\hat{h})\right]^{2}}{8\omega\hat{h}} \\ < \;& 0 \end{aligned}$$

That condition can be rewritten as:

$$\frac{\left(\Delta(\hat{h})\right)^{-1/2}\Delta'(\hat{h})}{2\hat{h}}+\delta\frac{\sqrt[2]{\Delta(\hat{h})}-\varphi}{\left(\hat{h}\right)^{3}}<\frac{\left(\Delta(\hat{h})\right)^{-3/2}\left[\Delta'(\hat{h})\right]^{2}}{8}\left(1+\frac{\delta}{\hat{h}}\right)$$

When that assumption is made, we know for sure that the threshold $\tilde{h}$, when it exists, is unique. Thus, once the economy has entered the Regime III, at which $\frac{\partial R_t}{\partial h_t}<0$, this remains within that regime for all future values of $h_t>\tilde{h}$.