OPTIMAL ECONOMIC GROWTH THROUGH CAPITAL ACCUMULATION IN A SPATIALLY HETEROGENEOUS ENVIRONMENT

RAOUF BOUCEKKINE*, GIORGIO FABBRI*, SALVATORE FEDERICO**, AND FAUSTO GOZZI***

Abstract. We design a general set-up for the study of a generic economy whose development process is entirely driven by the spatio-temporal dynamics of capital accumulation. It allows us to take into account spatial heterogeneities in technological level and population distribution. We solve analytically, via dynamic programming in infinite dimensions, the optimal control problem associated to the model, finding explicitly the optimal feedback and the value function. The expression of the optimal dynamics of the system in terms of eigenfunctions of an appropriate Sturm-Liouville problem allows to simulate the behavior of the variables and, in particular, their optimal discounted long-run spatial distribution.

Key words: Dynamic programming; dynamical spatial model; agglomeration; infinite dimensional optimal control problems, Sturm-Liouville theory.

Journal of Economic Literature Classification: R1; O4; C61.

1. Introduction

Arguably, modelling a spatial dimension is essential to understand a series of key features of economic development: distribution of economic activity, agglomeration of production and wealth, city formation, migrations, etc... so it is not surprising that the relation between development and space appears already among the interests of the classical economists (for instance Launhardt, von Thunen, Smith) and that space is at the core of the "fourth wave of the increasing-returns revolution in economics", as depicted in Fujita et al. (2001) who accurately describe the burst of the New Economic Geography in the last decade of the 20th century.

Given the importance of the subject it is more surprising that space was absent from economic growth models, and that the first attempt to introduce spatial capital mobility and accumulation in a modern economic growth

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*Aix-Marseille University (Aix-Marseille School of Economics), CNRS and EHESS..
**Università degli Studi di Siena, Dipartimento di Economia Politica e Statistica.
***Dipartimento di Economia e Finanza, LUISS *Guido Carli*, Roma.

setting only arrives with Brito (2004)[1]. In Brito's framework, the production is described at any spatial point as a neoclassical production function and employs local inputs; the output is used for *in situ* consumption and investment while the net trade flow depends on the differentials of the spatially distributed capital stock. This last hypothesis, consistent with recent empirical results of Comin et. al (2012), partly traces back to classical economic geographic literature of the 70s (see for instance the books of Isard and Liossatos, 1979 and of Beckmann and Puu, 1985) being the counterpart of models where "commodities flow from sources of excess supply to sinks of excess demand" (Ten Raa, 1986). This is also the element that drives the evolution of the economic system (via parabolic partial differential equations) in Brito's setting.

After the contribution of Brito (2004), several papers have developed and extended the initial setting and results. We refer the reader to Camacho et al. (2008), Boucekkine et al. (2009, 2013), Fabbri (2016), Balestra (2017) and the references mentioned there. The approach was also successfully applied to environmental management problems, see Brock and Xepapadeas (2015, 2016), Brock et al. (2014c), Camacho and Pérez-Barahona (2015) and La Torre et al. (2015).

Among the reasons of the late development of the spatial economic growth literature we can probably identify the technical difficulties of the problem. The fact that, as mentioned, the state equation of the involved optimal control problem is a parabolic PDE means, in particular, that the whole optimization problem is infinite dimensional. Specific difficulties arise, especially from the adjoint system as described for example by Boucekkine et al. (2009). Indeed several simplifying hypotheses have been used in various papers to be able to deal with the problem: for example, Brito (2004) focuson traveling waves solutions, Boucekkine al. (2009) give results for the case of linear utility and Brito (2011) studies the local dynamics of the system[2].

Boucekkine et al. (2013) are the first to solve explicitly a spatial growth model with capital mobility in the case of continuous spatial modeling[3]. They choose an AK production function and, as a spatial support, a circle à la Salop (1979), which avoids the problem of setting the most appropriate boundary condition. The proposed model is solved explicitly, the optimal solution is found in closed-form and the qualitative dynamics are closely described. Despite the intrinsic tendency toward divergence, typical of the endogenous growth models, the spatio-temporal dynamics of capital converge asymptotically to the unifom distribution.

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[1]Economic growth models with a spatial dimension were already formulated in the context of the New Economic Geography stream but, as observed by Desmet et Rossi-Hansberg (2010), they used to disregard intertemporal optimization behaviors and the capital accumulation process. Economic growth is typically modeled in the spirit of Grossman-Helpman and Aghion-Howitt, see for instance Nijkamp and Poot (1998).

[2]Another possible way to overcome a part of the technical difficulties is to take capital as fixed and only consider its possible spatial externalities. It is the idea used for instance by Brock et al. (2014a, 2014b) and Quah (2002).

[3]They solve the problem by using the dynamic programming in infinite dimensions. Recently the same problem is solved by Ballestra (2017) using an appropriate form of infinite dimensional maximum principle.

Indeed Boucekkine et al. (2013) study the simplest possible endogenous growth case and several elements of interest remain out of the picture. Maybe the most significant restriction from the point of view of the spatial representation is the hypothesis of complete neutrality of space: each point of the space is, from the intrinsic economic point of view, exactly equivalent to any other. Only the initial distribution of endowments differentiate them. This neutrality assumed is indeed the main reason behind the obtained uniform convergence result. In the present contribution we extend the model in two directions: (i) instead of supposing that technology is represented by an exogenous, time and space independent constant A, we allow for a space-heterogeneous distribution of the technology among locations; (ii) instead of hypothesizing a constant and uniformly distributed population, we are able to study the dynamics of the system by specifying any time-independent spatial distribution of the agents. As a particular case, considering uniform distributions for both technology and population leads exactly to Boucekkine et al 's model. We can therefore study the robustness of the asympotic convergence to uniform spatial distributions to population and technology space dependence.

The generality of the problem treated also improves in several respects the results of numerical works in the field. We refer the reader in particular to the paper by Camacho et al. (2008) which presents a computational study of a neoclassical spatial growth model *à la* Brito on the straight line *via* a maximum principle approach. To overcome the mentioned technical difficulty of studying the adjoint problem (in particular at infinity) they limit their attention to the finite horizon case and they only treat the case of an exponential spatial distribution of population. Avoiding these shortcuts has a certain importance: on the one hand the behavior of the finite-horizon model is qualitatively different from that of the infinite-horizon case (for instance the optimal capital at the finite terminal time needs to be 0 and this is odd for a growth model); on the other hand, it is potentially interesting to study the specific role of population distribution across space. In our work we can relax all these restrictions.

We are able to analytically solve the general case of the model by using the dynamic programming in infinite dimensions (developed in Section 4). Precisely, we are able to explicitly find the maximal welfare and the optimal consumption both in feedback form (i.e. in terms of the current distribution of wealth) and in explicit form. Ultimately, we can single out the PDE which delivers the optimal evolution of the spatiotemporal capital distribution and study the convergence properties considered. This result is obtained thanks to the main methodological novelty of the present work with respect to the existing literature in spatial growth models: the use of eigenfunctions of an appropriate Sturm-Liouville problem. This approach, that generalizes the Fourier-series expressions of Boucekkine et al. (2013) and those in terms of eigenfunctions of Laplace-Beltrami operators of Fabbri (2016), is used in the two main analytical results of the paper presented in Section 3: Theorem 3.2, where we characterize the optimal control of the problem in terms of the first eigenfunction of the (linear) zero-consumption problem; Theorem 3.3, where the long-run profile of the capital distribution is expressed as an infinite series of the eigenfunctions of the same operator. A precise description of the infinite dimensional techniques we use together with a complete proof of all the analytical results is given in Section 4.

In Section 5 we use the analytical results of Sections 3 and 4 to run some numerical simulations; we use the free software system *Chebfun*[4] written for MATLAB. The numerical results allow to quantify on an adequately calibrated version of the model, the two main effects at work when the space distributions of technology and population are heterogeneous. On the one hand we have the classical core-periphery effect: the planner has the incentive to favor the concentration of the capital in the areas where it is more productive so that she will tend to promote (relatively more) investment in areas where technological is better. On the other hand we have the population effect: the Benthamite form of the functional (that is the utility of each individual is weighted exactly in the same way, regardless of the position and of the population size in the location) induces the planner to guarantee an adequate level of per capita consumption across space so that areas with higher population get also a higher aggregate consumption and therefore a lower investment. The simulations of Section 5 show how the two effects work separately and then how they interact.

The paper proceeds as follows. Section 2 is devoted to description of the model. Section 3 presents the main analytical results. Section 4 provides the proofs of the analytical results via dynamic programming in infinite dimensions. Section 5 concerns numerical simulations and associated remarks. Section 6 concludes.

2. The model

We study a spatial economy developing on the unit circle S^1 in the plan[5]:

$$S^1 := \{(\sin\theta, \cos\theta) \in \mathbb{R}^2 \,:\, \theta \in [0, 2\pi)\}.$$

We suppose that, for all time $t \geq 0$ and any point in the space $\theta \in [0, 2\pi)$, the production is a linear function of the employed capital:

$$Y(t, \theta) = A(\theta)K(t, \theta),$$

where $K(t, \theta)$ and $Y(t, \theta)$ represent, respectively, the aggregate capital and output at the location θ at time t while $A(\theta)$ is the exogenous location-dependent technological level. In the model there is no state intervention and then, at any time, the local production is split into investment in local capital and local consumption so that, once we include a location-dependent

[4]The Chebfun system was initially introduced by Battles and Trefethen (2004), the modulus for eigenfunctions of Sturm-Liouville operators were originally conceived by Driscoll et al. (2008) and then implemented by Birkisson and Driscoll (2011) and Driscoll and Hale (2016).

[5]The functions over S^1 can be clearly identified with 2π-periodic functions over $\mathbb{R}$. We shall confuse these functions, as well as the point $\theta \in [0, 2\pi)$ with the corresponding point $(\sin\theta, \cos\theta) \in S^1$. Hence, given a function $f : S^1 \to \mathbb{R}$, the derivatives with respect to $\theta \in S^1$ will be intended through the identification of functions defined on S^1 with 2π-periodic functions defined on $\mathbb{R}$.

depreciation rate $\delta(\theta)$ and the net trade balance $\tau(t,\theta)$, we get the following accumulation law of capital:

$$\begin{aligned}\frac{\partial K}{\partial t}(t,\theta) &= I(t,\theta) - \delta(\theta)K(t,\theta) - \tau(t,\theta)\\ &= Y(t,\theta) - C(t,\theta) - \delta(\theta)K(t,\theta) - \tau(t,\theta)\\ &= (A(\theta) - \delta(\theta))K(t,\theta) - C(t,\theta) - \tau(t,\theta).\end{aligned}$$

We can always include the depreciation rate $\delta(\theta)$ in the coefficient $A(\theta)$ so the previous equation simply becomes

$$\frac{\partial K}{\partial t}(t,\theta) = A(\theta)K(t,\theta) - C(t,\theta) - \tau(t,\theta).$$

Following the idea of Brito (2004) (and then used by all the papers in the stream described in the introduction), given $0 \leq \theta_1 < \theta_2 < 2\pi$, the net trade balance over the region (θ_1, θ_2) is given by the balance of the flow of capital, at time t, at the boundaries θ_1 and θ_2:

$$\int_{\theta_1}^{\theta_2} \tau(t,\theta)d\theta = \frac{\partial K}{\partial \theta}(t,\theta_1) - \frac{\partial K}{\partial \theta}(t,\theta_2).$$

The last expression holds for any choice of θ_1 and θ_2 and it also equals the quantity $\int_{\theta_1}^{\theta_2} -\frac{\partial^2 K}{\partial \theta^2}(t,\theta)d\theta$ so, letting θ_2 to θ_1, we get, for any $\theta \in [0, 2\pi)$, $\tau(t,\theta) = -\frac{\partial^2 K}{\partial \theta^2}(t,\theta)$. The capital evolution law reads then as

$$\frac{\partial K}{\partial t}(t,\theta) = \frac{\partial^2}{\partial \theta^2}K(t,\theta) + A(\theta)K(t,\theta) - C(t,\theta).$$

If, for any (t,θ), we finally express the total consumption $C(t,\theta)$ as the product of the per-capita consumption[6] $c(t,\theta)$ and the time-independent exogenous (density of) population $N(\theta)$, we obtain

(1)
$$\begin{cases} \frac{\partial K}{\partial t}(t,\theta) = \frac{\partial^2}{\partial \theta^2}K(t,\theta) + A(\theta)K(t,\theta) - c(t,\theta)N(\theta), \quad t > 0,\, \theta \in S^1,\\ K(0,\theta) = K_0(\theta),\ \theta \in S^1. \end{cases}$$

where K_0 denotes the initial distribution of capital over the space S^1. We suppose that the policy maker operates to maximize the following intertemporal constant relative risk aversion functional:

$$\int_0^\infty e^{-\rho t} \int_0^{2\pi} \frac{c(t,\theta)^{1-\sigma}}{1-\sigma} N(\theta) d\theta dt, \tag{2}$$

where $\rho > 0$ and $\sigma \in (0,1) \cup (1,\infty)$ are given constant and the constraints $c(t,\theta) \geq 0$ and $K(t,\theta) \geq 0$ are imposed. This is indeed a Benthamite functional in the following sense: at any time t, the planner linearly weights the per-capita utility at any location using the population density. In other terms, the consumption/utility of all the people in the economy matter in the same way in the target. This fact will have a certain importance in the following.

The described model is a strict generalization of that considered by Boucekkine et al. (2013) because we consider here a technological level

[6]We suppose resources and consumption are equally distributed among the population of a certain location.

$A(\theta)$ and a population density $N(\theta)$ depending on the location θ. In other words here A and N are functions $A, N\colon S^1 \to \mathbb{R}$ instead of just two space-independent constants.

3. Main analytical results

In this section we present the two main analytical results of the paper that characterize the solution of the optimal control problem associated to the model described in the previous section and the corresponding behaviour of the system. As our results will be expressed in terms of the eigenvalues and the eigenfunctions of a suitable Sturm-Liouville problem, we begin our exposition by recalling the definitions of these concepts and some related results.

We consider the differential operator associated to the zero-consumption diffusion dynamics of (1), namely

$$\mathcal{L}u(\theta) := \frac{\partial^2}{\partial\theta^2}u(\theta) + A(\theta)u(\theta). \tag{3}$$

The operator $\mathcal{L}$ is well defined on regular enough functions $\phi\colon S^1 \to \mathbb{R}$. A non identically null regular function $\phi\colon S^1 \to \mathbb{R}$ is called *eigenfunction* of $\mathcal{L}$ if there exists a real number (*eigenvalue*) λ such that $\mathcal{L}\phi = \lambda\phi$. It can be proved (see Theorems 2.4.2 and 2.5.1 by Brown et al., 2013) that there is a countable discrete set of eigenvalues $\{\lambda_n\}_{n\geq 0}$ which can be ordered in decreasing way. The highest eigenvalue, λ_0, is associated to a unique eigenfunction (i.e. its multiplicity is 1) and this is the only eigenfunction without zeros. Eigenfunctions are defined up to a multiplicative factor; we denote by $\mathbf{e}_0$ the unique eigenfunction corresponding to the eigenvalue λ_0 such that $\mathbf{e}_0(\theta) > 0$ for each $\theta \in S^1$ and $\int_0^{2\pi} \mathbf{e}_0^2(\theta)d\theta = 1$. It can be proved (see again Theorems 2.4.2 and 2.5.1 of Brown et al., 2013) that the multiplicity of any other eigenvalue is either 1 or 2, that $\lambda_n \to -\infty$, as $n \to \infty$, and that there exists an orthonormal basis of $L^2(S^1)$ (see (11) for its definition) of eigenfunctions $\{\mathbf{e}_n\}_{n\geq 0}$ corresponding to the sequence of eigenvalues[7] $\{\lambda_n\}_{n\geq 0}$.

We have now collected the elements we need to describe the solution of the model and we can proceed by presenting it. We will work under the following spatial counterpart of the usual assumption on coefficient of the standard one-dimensional AK model to ensure the finiteness of the utility.

Hypothesis 3.1. *The discount rate satisfies*

$$\rho > \lambda_0(1 - \sigma). \tag{4}$$

The assumption that will make on A will imply that λ_0 is positive (see Remark 3.4). Hence, the previous condition is obviously verified when $\sigma > 1$ (that is the case for reasonable calibrations of the model, see Section 5).

[7] In the sequence $\{\lambda_n\}_{n\geq 0}$ a certain value appears once, respectively twice, if its multiplicity is 1, respectively 2.

Theorem 3.2. *Let Hypothesis 3.1 hold. Assume that $A, N\colon S^1 \to \mathbb{R}^+$ are bounded and not identically null, denote by α_0 the value*[8]

$$\alpha_0 := \left(\frac{\sigma}{\rho - \lambda_0(1-\sigma)} \int_0^{2\pi} \mathbf{e}_0(\theta)^{-\frac{1-\sigma}{\sigma}} \mathbf{N}(\theta) d\theta\right)^{\frac{\sigma}{1-\sigma}}, \tag{5}$$

and by $\boldsymbol{\beta}$ the function $\alpha_0 \mathbf{e}_0$.

Provided that the corresponding state trajectory remains positive, the control defined in feedback form (i.e. as a function of the capital distribution) as

$$c_K^*(\theta) = \left(\int_0^{2\pi} \boldsymbol{\beta}(\eta) K(\eta) d\eta\right) (\boldsymbol{\beta}(\theta))^{-1/\sigma}, \qquad \theta \in S^1, \tag{6}$$

is optimal, so that the optimal evolution of the capital density is given by the unique solution of the following PDE:

(7)

$$\begin{cases} \frac{\partial K}{\partial t}(t,\theta) = \frac{\partial^2}{\partial \theta^2} K(t,\theta) + A(\theta) K(t,\theta) - \left(\int_0^{2\pi} \boldsymbol{\beta}(\eta) K(\eta) d\eta\right) (\boldsymbol{\beta}(\theta))^{-1/\sigma} N(\theta) \\ K(0,\theta) = K_0(\theta), \ \theta \in S^1. \end{cases}$$

Moreover along the optimal trajectories the optimal consumption can also be expressed explicitly in terms of time and it is given by

$$c^*(t,\theta) = \left(\int_0^{2\pi} \boldsymbol{\beta}(\eta) K_0(\eta) d\eta\right) e^{gt} (\boldsymbol{\beta}(\theta))^{-1/\sigma},$$

where g is the growth rate of the economy, given by

$$g := \frac{\lambda_0 - \rho}{\sigma}. \tag{8}$$

Once we compare the optimal consumption profile described in the previous theorem with that of Boucekkine et al. (2013) we can immediately see the importance of the possibility of using a location-dependent coefficient A. Indeed in the setting of Boucekkine et al. (2013) the (per-capita and aggregate) optimal consumption level is always equal among the locations while here the expression of the optimal consumption is given by the space-independent term $\left(\int_0^{2\pi} \boldsymbol{\beta}(\eta) K_0(\eta) d\eta\right) e^{gt}$ and by the space-dependent term $(\boldsymbol{\beta}(\theta))^{-1/\sigma} = (\alpha_0 \mathbf{e}_0)^{-1/\sigma}$. The latter depends on $A(\cdot)$ both *via* α_0 and $\mathbf{e}_0$ and on $N(\cdot)$ *via* α_0. This fact is interesting from a theoretical point of view since *a priori* one might guess that the egalitarian character of the Benthamite functional could be enough to guarantee a spatial equal individual utility. On the contrary the structural conditions of the economy can suggest to the planner to diversify the per-capita consumption among locations. As we will see in Section 5 the differentiation does not always go in the expected way.

The difference with Boucekkine et al. (2013) is also very significant in our second result, describing the long-run profile of the detrended optimal capital: while in case of space-constant A and N the space-distribution of the wealth always converges (under the hypotheses of Theorem 3.3) to a uniform profile, here an articulated expression, depending on the whole technological and human population distributions, arises.

[8] This number is well defined and strictly positive thanks to (4).

Theorem 3.3. *Let hypotheses of Theorem 3.2 hold and suppose that*

$$g > \lambda_1 \tag{9}$$

where g is defined in (8) and λ_1 is the second eigenvalue of the problem considered above. Define the detrended optimal path $K_g(t,\theta) := e^{-gt}K(t,\theta)$, for $t \geq 0$. Then

$$K_g(t,\theta) \overset{t\to\infty}{\longrightarrow} \int_0^{2\pi} K_0(\eta)\boldsymbol{\beta}(\eta)d\eta \left(\frac{\mathbf{e}_0(\theta)}{\alpha_0} + \sum_{n\geq 1} \frac{\beta_n}{\lambda_n - g}\mathbf{e}_n(\theta) \right)$$

where, for $n \geq 1$,

$$\beta_n := \int_0^{2\pi} (\boldsymbol{\beta}(\eta))^{-1/\sigma} N(\eta)\mathbf{e}_n(\eta)d\eta.$$

Remark 3.4. *The following estimates on λ_0 can be obtained from its representation provided in Section 2.10 of Brown et al. (2013):*

$$\frac{1}{2\pi}\int_0^{2\pi} A(\theta)d\theta \ \leq\ \lambda_0 \ \leq\ \sup_{S^1}|A|. \tag{10}$$

The lower bound in particular assures, given the positivity of $A(\cdot)$, the positivity of λ_0. The upper bound is useful to check (4),

Theorem 2.9.3 of Brown et al. (2013) also gives the following estimates for the second eigenvalue:

$$\lambda_1 \ \leq \sup_{S^1} A - 1,$$

useful to check (9).

4. Proofs of the analytical results

4.1. **The infinite dimensional setting.** We can represent (1) as an abstract dynamical system in infinite-dimension. Some step is needed to describe this constructions. Consider the space

$$\mathcal{H} := L^2(S^1) := \left\{ f : S^1 \to \mathbb{R} \text{ measurable} \ \Big|\ \int_0^{2\pi} |f(\theta)|^2 d\theta < \infty \right\}. \tag{11}$$

This is a Hilbert space when endowed with the inner product $\langle f, g\rangle := \int_0^{2\pi} f(\theta)g(\theta)d\theta$, inducing the norm $\|f\| = \int_0^{2\pi} |f(\theta)|^2 d\theta$. We will also use the following spaces of real functions defined on S^1:

$$L^\infty(S^1) := \{f \in \mathcal{H} \mid |f| \leq C \text{ for some } C > 0\},$$
$$H^1(S^1) := \{f \in \mathcal{H} \mid \exists\ f' \text{ in weak sense and belongs to } \mathcal{H}\},$$
$$H^2(S^1) := \{f \in \mathcal{H} \mid \exists\ f' \text{ in weak sense and belong to } H^1(S^1)\}.$$

Suppose from now that that the coefficients of the state equation satisfy the following conditions:

$$A \in L^\infty(S^1), \qquad N \in L^\infty(S^1). \tag{12}$$

The differential operator

$$\mathcal{L}u := \frac{\partial^2 u}{\partial\theta^2} + A(\cdot)u, \qquad u \in H^2(S^1)$$

is well defined and $\mathcal{H}$-valued. It is also self-adjoint, i.e.

$$\mathcal{L}^* = \mathcal{L}. \tag{13}$$

The operator $\mathcal{L}$ is the sum of the Laplacian operator on S^1 with the bounded operator $\mathbf{A} : \mathcal{H} \to \mathcal{H},\ u \mapsto A(\cdot)u$. The Laplacian operator is closed on the domain $H^2(S^1)$ and generates a C_0-semigroup on the space $\mathcal{H}$. Hence, as $\mathbf{A}$ is bounded, we deduce that also $\mathcal{L}$ is closed on the domain

$$D(\mathcal{L}) := H^2(S^1)$$

and generates a C_0-semigroup on the space $\mathcal{H}$. From now on, in order to avoid confusion, we will denote the elements of $\mathcal{H}$ by bold letters. With this convention, we can formally rewrite (1) as an abstract dynamical system in the space $\mathcal{H}$:

$$\begin{cases} \mathbf{K}'(t) = \mathcal{L}\mathbf{K}(t) - \mathbf{c}(t)\mathbf{N}, & t \in \mathbb{R}^+, \\ \mathbf{K}(0) = \mathbf{K}_0 \in \mathcal{H}, \end{cases} \tag{14}$$

with the formal equalities $\mathbf{K}(t)(\theta) = K(t,\theta)$, $[\mathbf{c}(t)\mathbf{N}](\theta) = c(t,\theta)N(\theta)$ and we will read the original system as (14).[9]

By general theory of semigroups (see Proposition 3.1 and 3.2, Section II-1, of Bensoussan et al., 2007, also considering (13)), given $\mathbf{c} \in L^1_{loc}(\mathbb{R}^+;\mathcal{H})$, there exists a unique (weak) solution $\mathbf{K}^{\mathbf{K}_0,\mathbf{c}} \in L^1_{loc}(\mathbb{R}^+;\mathcal{H})$ to (14) in the following sense: for each $\boldsymbol{\varphi} \in D(\mathcal{L})$ the function $t \mapsto \langle \mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t), \boldsymbol{\varphi}\rangle$ is locally absolutely continuous and

$$\begin{cases} \frac{d}{dt}\langle \mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t), \boldsymbol{\varphi}\rangle = \langle \mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t), \mathcal{L}\boldsymbol{\varphi}\rangle - \langle \mathbf{c}(t)\mathbf{N}, \boldsymbol{\varphi}\rangle, & a.e.\ t \in \mathbb{R}^+, \\ \mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(0) = \mathbf{K}_0 \in \mathcal{H}. \end{cases} \tag{15}$$

Consider the positive cone in $\mathcal{H}$, i.e. the set

$$\mathcal{H}^+ := \{\mathbf{K} \in \mathcal{H} \mid \mathbf{K}(\cdot) \geq 0\},$$

the positive cone in $\mathcal{H}$ without the null function, i.e. the set

$$\mathcal{H}^+_0 := \{\mathbf{K} \in \mathcal{H} \mid \mathbf{K}(\cdot) \geq 0 \text{ and } \mathbf{K}(\cdot) \not\equiv 0\},$$

and define the set of admissible strategies as[10]

$$\mathcal{A}(\mathbf{K}_0) := \{\mathbf{c} \in L^1_{loc}(\mathbb{R}^+;\mathcal{H}^+) \mid \mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t) \in \mathcal{H}^+_0 \quad \forall t \geq 0\}.$$

Then we can rewrite the original optimization problem as

$$(\mathbf{P}) \qquad V(\mathbf{K}_0) := \sup_{\mathbf{c}\in\mathcal{A}(\mathbf{K}_0)} J(\mathbf{K}_0;\mathbf{c}), \tag{16}$$

[9]The correspondence between the concept of solution to the abstract dynamical system in $\mathcal{H}$ that we introduce below (weak solution) and the solution of can be argued as in Proposition 3.2, page 131, of Bensoussan et al. (2007).

[10]In this formulation we require the slightly sharper state constraint $\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t) \in \mathcal{H}^+_0$ in place of the wider (original) one $\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t)(\cdot) \geq 0$ almost everywhere. This is without loss of generality: indeed, if $\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t) \equiv 0$ at some $t \geq 0$, the unique admissible (hence the optimal) control from t on is the trivial one $\mathbf{c}(\cdot) \equiv 0$, so we know how to solve the problem once we fall into this state and there is no need to define the Hamilton-Jacobi-Bellman equation at this point. The reason to exclude the null function from the set $\mathcal{H}^+$ and considering the set $\mathcal{H}^+_0$ is allowing a well-definition of the Hamilton-Jacobi-Bellman equation.

where

$$J(\mathbf{K}_0;\mathbf{c}) := \int_0^\infty e^{-\rho t}\mathcal{U}(\mathbf{c}(t))dt,$$

and

$$\mathcal{U}:\mathcal{H}^+ \to \mathbb{R}^+,\ \ \mathcal{U}(\mathbf{c}) := \int_0^{2\pi} \frac{\mathbf{c}(\theta)^{1-\sigma}}{1-\sigma}\mathbf{N}(\theta)d\theta.$$

4.2. **HJB equation.** The Hamilton-Jacobi-Bellman (HJB) equation in $\mathcal{H}$ associated to (16) is defined as follows

$$\rho v(\mathbf{K}) = \langle \mathbf{K}, \mathcal{L}\nabla v(\mathbf{K})\rangle + \sup_{\mathbf{c}\in\mathcal{H}^+} \{\mathcal{U}(\mathbf{c}) - \langle \mathbf{cN}, \nabla v(\mathbf{K})\rangle\}. \tag{17}$$

An explicit solution of this equation can be given in a suitable half-space of $\mathcal{H}$ as shown by the following proposition.

Proposition 4.1. *Let (4) and (12) hold. The function*

$$v(\mathbf{K}) = \frac{\langle \mathbf{K}, \alpha_0\mathbf{e}_0\rangle^{1-\sigma}}{1-\sigma}, \quad \mathbf{K}\in\mathcal{H}^+_{\mathbf{e}_0}, \tag{18}$$

where

$$\mathcal{H}^+_{\mathbf{e}_0} := \{\mathbf{K}\in\mathcal{H} \mid \langle \mathbf{K},\mathbf{e}_0\rangle > 0\}. \tag{19}$$

and

$$\alpha_0 := \left(\frac{\sigma}{\rho-\lambda_0(1-\sigma)}\int_0^{2\pi} \mathbf{e}_0(\theta)^{-\frac{1-\sigma}{\sigma}}\mathbf{N}(\theta)d\theta\right)^{\frac{\sigma}{1-\sigma}}, \tag{20}$$

solves (17) *over* $\mathcal{H}^+_{\mathbf{e}_0}$.

Proof. Define the strictly positive cone in $\mathcal{H}$, i.e.

$$\mathcal{H}^{++} := \left\{ f : S^1 \to \mathbb{R}^{++} \ \Big|\ \int_0^{2\pi} |f(\theta)|^2 d\theta < \infty \right\},$$

Setting

$$\mathcal{U}^*(\boldsymbol{\alpha}) := \sup_{\mathbf{c}\in\mathcal{H}^+} \{\mathcal{U}(\mathbf{c}) - \langle \mathbf{cN}, \boldsymbol{\alpha}\rangle\}, \quad \boldsymbol{\alpha}\in\mathcal{H}^{++},$$

we have

$$\mathcal{U}^*(\boldsymbol{\alpha}) := \sup_{\mathbf{c}\in\mathcal{H}^+} \int_0^{2\pi} \left(\frac{\mathbf{c}(\theta)^{1-\sigma}}{1-\sigma}\mathbf{N}(\theta) - \mathbf{c}(\theta)\mathbf{N}(\theta)\boldsymbol{\alpha}(\theta)\right)d\theta = \int_0^{2\pi} u^*(\boldsymbol{\alpha}(\theta))d\theta,$$

where

$$u^*(q) := \sup_{c\geq 0}\left\{\frac{c^{1-\sigma}}{1-\sigma}N - qcN\right\} = \frac{\sigma}{1-\sigma}Nq^{-\frac{1-\sigma}{\sigma}}, \quad q>0,\ N\geq 0,$$

with optimizer

$$c^*(q) = q^{-\frac{1}{\sigma}}, \quad q>0. \tag{21}$$

Plugging (18) into (17), we need to check the equality

$$\frac{\rho}{1-\sigma}\langle \mathbf{K},\alpha_0\mathbf{e}_0\rangle^{1-\sigma} = \langle \mathbf{K}, \mathcal{L}\alpha_0\mathbf{e}_0\rangle\langle \mathbf{K},\mathbf{e}_0\rangle^{-\sigma} + \frac{\sigma}{1-\sigma}\left(\int_0^{2\pi}\alpha_0^{-\frac{1-\sigma}{\sigma}}\mathbf{e}_0(\theta)^{-\frac{1-\sigma}{\sigma}}\mathbf{N}(\theta)d\theta\right)\langle \mathbf{K},\alpha\mathbf{e}_0\rangle^{1-\sigma}. \tag{22}$$

By definition of λ_0 and $\mathbf{e}_0$, we have $\mathcal{L}\mathbf{e}_0 = \lambda_0\mathbf{e}_0$. So (22) holds by (20). □

For notational reasons we set

$$\boldsymbol{\beta} := \alpha_0 \mathbf{e}_0,$$

so we can rewrite (18) as

$$v(\mathbf{K}) = \frac{\langle \mathbf{K}, \boldsymbol{\beta}\rangle^{1-\sigma}}{1-\sigma}, \quad \mathbf{K} \in \mathcal{H}^+_{\mathbf{e}_0}. \tag{23}$$

Finally, from the definition of $\boldsymbol{\beta}$ and (20) we get the following identity that will be useful in the next subsection

$$\left(\int_0^{2\pi} \boldsymbol{\beta}(\theta)^{-\frac{1-\sigma}{\sigma}} \mathbf{N}(\theta) d\theta\right) = \frac{\rho - \lambda_0(1-\sigma)}{\sigma}. \tag{24}$$

4.3. **Solution of the optimal control problem via dynamic programming in infinite dimensions.** Proposition 4.1 suggests to consider a different set of admissible controls, i.e.

$$\mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0) := \{\mathbf{c} \in L^1_{loc}(\mathbb{R}^+; \mathcal{H}^+) \mid \mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t) \in \mathcal{H}^+_{\mathbf{e}_0} \quad \forall t \geq 0\}.$$

Since $\mathcal{H}^+_0 \subseteq \mathcal{H}^+_{\mathbf{e}_0}$, we have also $\mathcal{A}(\mathbf{K}_0) \subseteq \mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0)$. We define an auxiliary problem associated to this new relaxed constraint, i.e.

$$(\tilde{\mathbf{P}}) \qquad \tilde{V}(\mathbf{K}_0) := \sup_{\mathbf{c} \in \mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0)} J(\mathbf{K}_0; \mathbf{c}). \tag{25}$$

Clearly we have the inequality

$$\tilde{V} \geq V \quad \text{over } \mathcal{H}^+_{\mathbf{e}_0}. \tag{26}$$

The reason to consider the relaxed state constraint $\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(\cdot) \in \mathcal{H}^+_{\mathbf{e}_0}$, in place of the stricter original one $\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(\cdot) \in \mathcal{H}^+_0$, is that the former is somehow the natural one from the mathematical point of view and allows a natural solution. On the other hand, the real constraint is still $\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(\cdot) \in \mathcal{H}^+$, so we need to establish a relationship between the two problems $(\mathbf{P})$ and $(\tilde{\mathbf{P}})$. Our approach relies on the following obvious result.

Lemma 4.2. *If $\mathbf{c}^*$ is an optimal control for $(\mathbf{P})$ and $\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(\cdot) \in \mathcal{H}^+_0$ (i.e. the solution of the optimization problem with relaxed state constraint actually satisfies the stricter one), then $\mathbf{c}^*$ is optimal also for $(\mathbf{P})$.*

We focus on the solution to $(\tilde{\mathbf{P}})$. Considering (21), the feedback map associated to the function v defined in (23) results in

$$\mathcal{H}^+_{\mathbf{e}_0} \to \mathcal{H}^+_0, \quad \mathbf{K} \mapsto \langle \boldsymbol{\beta}, \mathbf{K}\rangle \boldsymbol{\beta}^{-\frac{1}{\sigma}}, \tag{27}$$

where $\boldsymbol{\beta}^{-\frac{1}{\sigma}}(\theta) := (\boldsymbol{\beta}(\theta))^{-\frac{1}{\sigma}}$. The associated closed loop equation

$$\begin{cases} \mathbf{K}'(t) = \mathcal{L}\mathbf{K}(t) - \langle \boldsymbol{\beta}, \mathbf{K}(t)\rangle \boldsymbol{\beta}^{-\frac{1}{\sigma}} \mathbf{N} = \mathcal{L}\mathbf{K}(t) - \langle \boldsymbol{\beta}, \mathbf{K}(t)\rangle \boldsymbol{\beta}^{-\frac{1}{\sigma}} \mathbf{N}, \\ \mathbf{K}(0) = \mathbf{K}_0 \in \mathcal{H}^+_0, \end{cases} \tag{28}$$

admits a unique weak solution, i.e. there exists a unique function $\mathbf{K}^{\mathbf{K}_0,*} \in L^1_{loc}(\mathbb{R}^+; \mathcal{H})$ such that the function $t \mapsto \langle \mathbf{K}^{\mathbf{K}_0,*}(t), \boldsymbol{\varphi}\rangle$ is absolutely continuous for every $\boldsymbol{\varphi} \in D(\mathcal{L})$ and

(29)

$$\begin{cases} \frac{d}{dt}\langle \mathbf{K}^{\mathbf{K}_0,*}(t), \boldsymbol{\varphi}\rangle = \langle \mathbf{K}^{\mathbf{K}_0,*}(t), \mathcal{L}\boldsymbol{\varphi}\rangle - \langle \boldsymbol{\beta}, \mathbf{K}^{\mathbf{K}_0,*}(t)\rangle \langle \boldsymbol{\varphi}, \boldsymbol{\beta}^{-\frac{1}{\sigma}} \mathbf{N}\rangle, \quad a.e.\ t \in \mathbb{R}^+, \\ \mathbf{K}^{\mathbf{K}_0,*}(0) = \mathbf{K}_0 \in \mathcal{H}^+_0. \end{cases}$$

Consider (24) and set

$$g := \lambda_0 - \int_0^{2\pi} \mathbf{N}(\theta)\boldsymbol{\beta}(\theta)^{-\frac{1-\sigma}{\sigma}} d\theta = -\frac{\rho - \lambda_0}{\sigma}. \tag{30}$$

Taking $\boldsymbol{\varphi} = \boldsymbol{\beta}$ in (29), we get

$$\langle \mathbf{K}^{\mathbf{K}_0,*}(t), \boldsymbol{\beta} \rangle = \langle \boldsymbol{\beta}, \mathbf{K}_0 \rangle e^{gt}, \qquad t \geq 0, \tag{31}$$

Hence

$$\mathbf{K}_0 \in \mathcal{H}^+_{\mathbf{e}_0} \;\Rightarrow\; \mathbf{K}^{\mathbf{K}_0,*}(t) \in \mathcal{H}^+_{\mathbf{e}_0}.$$

So the control

$$\mathbf{c}^*(t) := \langle \boldsymbol{\beta}, \mathbf{K}(t) \rangle \boldsymbol{\beta}^{-\frac{1}{\sigma}} = \langle \boldsymbol{\beta}, \mathbf{K}_0 \rangle \boldsymbol{\beta}^{-\frac{1}{\sigma}} e^{gt}, \quad t \geq 0, \tag{32}$$

belongs to $\mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0)$.

Lemma 4.3. *For each $\mathbf{c} \in \mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0)$ we have*

$$\langle \mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t), \boldsymbol{\beta} \rangle \leq \langle \boldsymbol{\beta}, \mathbf{K}_0 \rangle e^{\lambda_0 t}, \quad \forall t \geq 0.$$

Proof. Denote by $\mathbf{0}$ the null control, i.e. the control $\mathbf{c}(t)(\theta) = 0$ for each $(t,\theta) \in \mathbb{R}^+ \times S^1$. Then (15) yields $\langle \mathbf{K}^{\mathbf{K}_0,\mathbf{0}}(t), \boldsymbol{\beta} \rangle = \langle \boldsymbol{\beta}, \mathbf{K}_0 \rangle e^{\lambda_0 t}$ for every $t \geq 0$. On the other hand, as $\boldsymbol{\beta}(\theta) > 0$ for each $\theta \in S^1$, standard comparison applied to the ODE (15) yields

$$\langle \mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(\cdot), \boldsymbol{\beta} \rangle \leq \langle \mathbf{K}^{\mathbf{K}_0,\mathbf{0}}(\cdot), \boldsymbol{\beta} \rangle, \tag{33}$$

and the claim follows. □

Theorem 4.4. *Let (4) and (12) hold. Let $\mathbf{K}_0 \in \mathcal{H}^+_{\mathbf{e}_0}$ and let v be the function defined in* (23)*. Then $v(\mathbf{K}_0) = \tilde{V}(\mathbf{K}_0)$ and the control $\mathbf{c}^*$ defined in* (32) *is optimal for* ($\tilde{\mathbf{P}}$) *starting from the initial state $\mathbf{K}_0$; i.e. $J(\mathbf{K}_0; \mathbf{c}^*) = \tilde{V}(\mathbf{K}_0)$.*

Proof. The fact that $\mathbf{c}^* \in \mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0)$ has been already observed. We prove now the optimality. By the usual arguments employed to prove Verification Theorem with a Dynamic Programming approach, using the fact that v is a solution to (17) on $\mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0)$ one gets, for every $\mathbf{c} \in \mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0)$,

$$\begin{aligned} e^{-\rho t} v(\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(t)) - v(\mathbf{K}_0) &= -\int_0^t e^{-\rho s} \mathcal{U}(\mathbf{c}(s)) ds \\ &+ \int_0^t e^{-\rho s} \{ \mathcal{U}(\mathbf{c}(s)) - \langle \mathbf{c}(s)\mathbf{N}, \nabla v(\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(s)) \rangle - \mathcal{U}^*(\nabla v(\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(s)) \} ds \end{aligned} \tag{34}$$

We pass (34) to the limit for $t \to \infty$.

- We use (4) and Lemma 4.3 in the left hand side;
- we use monotone convergence in the right hand side, as, by definition of $\mathcal{U}^*$, the integrand is nonpositive.

So, we get the so called *fundamental identity*, valid for each $\mathbf{c} \in \mathcal{A}^+_{\mathbf{e}_0}(\mathbf{K}_0)$:

$$\begin{aligned} v(\mathbf{K}_0) &= J(\mathbf{K}_0; \mathbf{c}) \\ &+ \int_0^\infty e^{-\rho s} \big\{ \mathcal{U}^*(\nabla v(\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(s)) - \big(\mathcal{U}(\mathbf{c}(s)) - \langle \mathbf{c}(s)\mathbf{N}, \nabla v(\mathbf{K}^{\mathbf{K}_0,\mathbf{c}}(s)) \rangle \big) \big\} ds. \end{aligned} \tag{35}$$

From (35), by definition of $\mathcal{U}^*$ we first get $v(\mathbf{K}_0) \geq \tilde{V}(\mathbf{K}_0)$. Then, observing that the integrand in (35) vanishes when $\mathbf{c} = \mathbf{c}^*$, we obtain $v(\mathbf{K}_0) = J(\mathbf{K}_0; \mathbf{c}^*)$. The claim follows. □

From Theorem 4.4 and Lemma 4.2, we get our first main result corresponding to Theorem 3.2.

Corollary 4.5. *Let (4) and (12) hold. Let $\mathbf{K}_0 \in \mathcal{H}_0^+$, let $\mathbf{c}^*$ be the control defined in* (32) *and assume that $\mathbf{c}^* \in \mathcal{A}(\mathbf{K}_0)$. Then $v(\mathbf{K}_0) = V(\mathbf{K}_0)$ and $\mathbf{c}^*$ is optimal for* (**P**).

The study the convergence of the transitional dynamics to a stationary state gives the following claim corresponding to Theorem 3.3.

Proposition 4.6. *Let (4), (9) and (12) hold. Define the detrended optimal path*

$$\mathbf{K}_g^{\mathbf{K}_0,\mathbf{c}^*}(t) := e^{-gt}\mathbf{K}^{\mathbf{K}_0,\mathbf{c}^*}(t), \qquad t \geq 0.$$

Then

$$\mathbf{K}_g^{\mathbf{K}_0,\mathbf{c}^*}(t) \overset{t\to\infty}{\longrightarrow} \langle \mathbf{K}_0, \boldsymbol{\beta}\rangle \left(\alpha_0^{-1}\mathbf{e}_0 + \sum_{n\geq 1} \frac{\beta_n}{\lambda_n - g}\mathbf{e}_n \right), \ \textit{in } L^2(S^1),$$

where $\beta_n := \langle \mathbf{e}_n, \boldsymbol{\beta}^{-\frac{1}{\sigma}}\mathbf{N}\rangle$ for $n \geq 1$.

Proof. As $\mathbf{K}^{\mathbf{K}_0,\mathbf{c}^*}(\cdot)$ is a weak solution of (28), $\mathbf{K}_g^{\mathbf{K}_0,\mathbf{c}^*}(\cdot)$ is a weak solution of

$$\begin{cases} \mathbf{K}'(t) = \mathcal{L}\mathbf{K}(t) - g\mathbf{K}(t) - \langle \boldsymbol{\beta}, \mathbf{K}(t)\rangle \boldsymbol{\beta}^{-\frac{1}{\sigma}}\mathbf{N} \\ \mathbf{K}(0) = \mathbf{K}_0 \in \mathcal{H}_0^+, \end{cases}$$

i.e., for every $\boldsymbol{\varphi} \in D(\mathcal{L})$,

(36)

$$\begin{cases} \frac{d}{dt}\langle \mathbf{K}_g^{\mathbf{K}_0,\mathbf{c}^*}(t), \boldsymbol{\varphi}\rangle = \langle \mathbf{K}_g^{\mathbf{K}_0,\mathbf{c}^*}(t), (\mathcal{L} - g)\boldsymbol{\varphi}\rangle - \langle \boldsymbol{\beta}, \mathbf{K}_g^{\mathbf{K}_0,\mathbf{c}^*}(t)\rangle \langle \boldsymbol{\varphi}, \boldsymbol{\beta}^{-\frac{1}{\sigma}}\mathbf{N}\rangle \\ \mathbf{K}_g^{\mathbf{K}_0,\mathbf{c}^*}(0) = \mathbf{K}_0 \in \mathcal{H}_0^+. \end{cases}$$

As already recalled in Section 3 there exists an orthonormal basis of $L^2(S^1)$ of eigenfunctions $\{\mathbf{e}_n\}_{n\geq 0}$ corresponding to the sequence of eigenvalues $\{\lambda_n\}_{n\geq 0}$ so we have the Fourier series expansion

$$\mathbf{K}_g^{\mathbf{K}_0,\mathbf{c}^*}(t) = \sum_{n\geq 0} K_{g,n}(t)\mathbf{e}_n, \quad \text{where } K_{g,n}(t) := \langle \mathbf{K}_g^{\mathbf{K}_0,\mathbf{c}^*}(t), \mathbf{e}_n\rangle, \qquad n \geq 0.$$

We compute now the Fourier coefficients $K_{g,n}(t)$.

- When $n = 0$, we already know from (31)

$$K_{g,0}(\cdot) \equiv \langle \mathbf{K}_0, \mathbf{e}_0\rangle = \alpha_0^{-1}\langle \mathbf{K}_0, \boldsymbol{\beta}\rangle.$$

- When $n \geq 1$, we have

$$K'_{g,n}(t) = (\lambda_n - g)K_{g,n}(t) - \langle \mathbf{K}_0, \boldsymbol{\beta}\rangle \beta_n.$$

So, we can explicitly express the Fourier coefficients as:

$$K_{g,n}(t) = \langle \mathbf{K}_0, \mathbf{e}_n\rangle e^{(\lambda_n - g)t} + \langle \mathbf{K}_0, \boldsymbol{\beta}\rangle \frac{\beta_n}{\lambda_n - g}(1 - e^{(\lambda_n - g)t}).$$

Considering that $\lambda_n \leq \lambda_1 < g$ for every $n \geq 1$, we have the convergence

$$K_{g,n}(t) \overset{t\to\infty}{\longrightarrow} \langle \mathbf{K}_0, \boldsymbol{\beta}\rangle \frac{\beta_n}{\lambda_n - g}, \qquad \text{uniformly in } n \geq 1.$$

The claim follows. □

5. Numerical analysis

The explicit representation of the long-run configuration of the economy given in Theorem 3.3 can be used to undertake a numerical analysis of the system in some specific cases of interest. To numerically compute the eigenfunctions e_n we use the package *Chebfun*[11] written for MATLAB.

First we calibrate the model using realistic values. In all the simulations we choose the discounting parameter ρ equal to 3% (consistent e.g. with the data of Lopez, 2008) and the inverse of the elasticity of intertemporal substitution σ equal to 5 (here it is also the constant relative risk aversion of the utility function so its value is coherent with those found e.g. by Barsky et al., 1997). In all our simulations we use the non-uniform technological distribution $A(\cdot)$ on $[0, 2\pi]$ having a pick at the point π (the "core") and attaining lower values in the further locations (the "periphery") represented in the first picture of Figure 1. The values of $1/A$ (that is the value of the ratio capital-over-output K/Y that in the model also equals the wealth-over-GDP ratio) is in the range $4 \div 6$ in line with the values found by Piketty and Zucman (2014).

In the described situation, computing the first eigenvalue of the operator $\mathcal{L}$ defined in (3) and using (8) we get the reasonable value of the global growth rate equal to 3.17%. As a further check we also observe that the (spatial-heterogeneous) saving rate in the long-run varies from 18% to 37% in line for instance with the World Bank data (see e.g. World Bank Group, 2016).

The effect of this non-uniform spatial technological distribution, whenever the population is constant with density everywhere equal to 1, is represented in Figure 1. We can promptly see the effect of the spatial polarization of the capital marginal (and average) productivity on capital accumulation in the first picture of the second line of Figure 1. In fact the capital tends to accumulate at the core where it is more productive while areas with smaller technology level remain behind: the higher productivity of capital in the core locations pushes the planner to increase investments and thus savings relatively more in these regions as shown in the second picture of the third line of Figure 1. As a byproduct the planner privileges consumption in peripheral regions but this is a second-order effect of small magnitude as one can see in the first picture of the third line of Figure 1.

Looking at the (spatial) relative magnitudes in the distributions of A and of the long-run detrended K, we can easily realize that the capital distribution is much less concentrated than the technological level[12]. We have indeed an endogenous spatial spillover effect that is the combined result both of the capital exogenous diffusivity and the endogenous investment and consumption decisions by the planner.

[11] See Birkisson and Driscoll (2011) and Driscoll and Hale (2016) for details on the implementation of the routines on linear differential operators and in particular on eigenfunctions of Sturm-Liouville operators in Chebfun.

[12] Conversely the concentration of the long-run detrended output is more picked because the output has the form $Y = AK$.

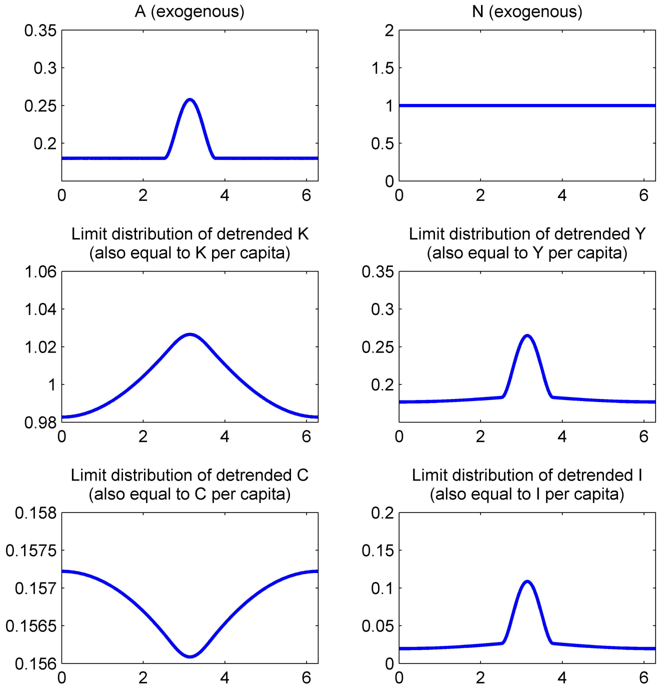

FIGURE 1. Core-periphery effect

The difference with respect to the results of Boucekkine at al. (2013) is crystal clear: once we introduce the spatial heterogeneity in capital productivity, the optimal detrended capital does not converge anymore to a spatial-homogeneous distribution. Indeed the situation described by Boucekkine at al. (2013), where all the detrended variables (capital, output, consumption, investment) converge to the spatial-homogeneous configuration, arises as a special case, only if A is constant over the locations. The behaviour confirms the intuition of Camacho at al. (2008) in the case of neoclassical production function. Indeed, even if in that work "the authors cannot prove the existence nor the uniqueness of the steady state solution to problem", their simulations in Sections 4.1 and 4.2 suggest that in the long-run the system converges towards a constant distribution only if A is constant across space.

Figure 2 emphasizes the pure dilution effect we have in the model. We consider the same technological distribution as in the previous picture and we vary uniformly the population density, more precisely we double the previous constant population density (in the picture the previous benchmark situation is in blue, with continuous line, while the new profile is in red, dotted line). The effect, in terms of aggregate optimal behavior is zero while per-capita variables are mechanically halved. This effect could be predicted directly from expression (6) taking into account the effect of population distribution on α given by (5). Observe that the pure dilution effect is not

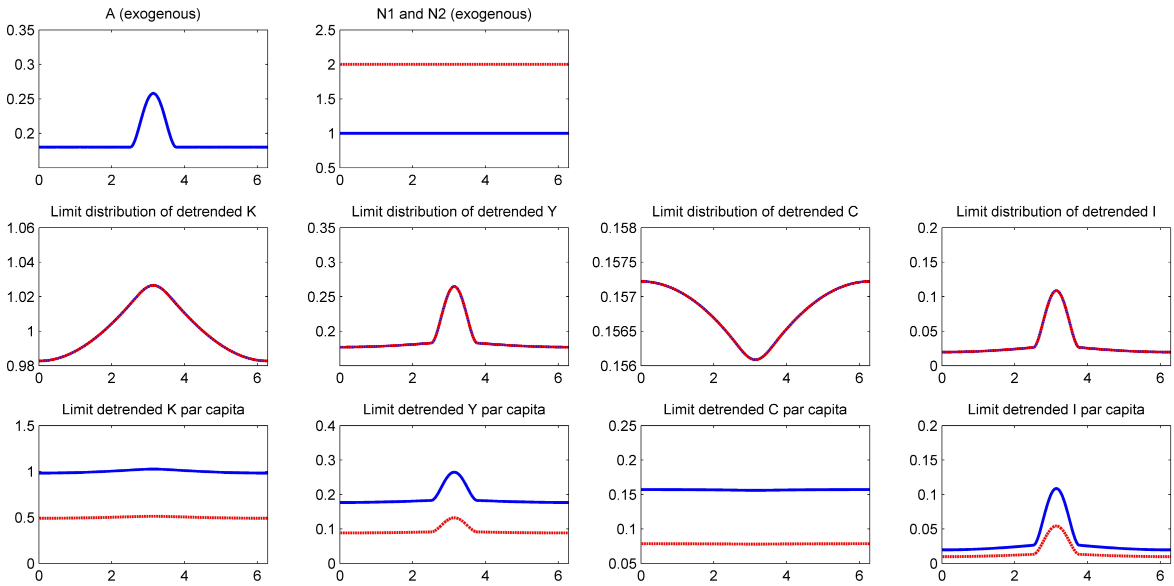

FIGURE 2. The pure dilution effect

due to the homogeneous distribution of the population we use: whatever the initial population distribution, a uniform increase of the population of $n\%$ in the whole space induces a spatial uniform proportional reduction (by a factor $\frac{1}{1+n/100}$) of per capita variables.

The dilution effect also appears in the finite horizon neoclassical spatial version of the model studied by of Camacho et al. (2008) (see in particular Section 4.2) but in their case the interaction between population distribution and capital accumulation decisions is partially driven by the factors' decreasing returns typical of the Cobb-Douglas production function.

In Figure 3 we consider a concentration of capital productivity and population density in the same areas (a quite frequent configuration) showing how the core-periphery and the population effects combine and can partially offset each other. In the simulation we keep the same technology distribution as before and we consider two possible population distributions: in the first one (the blue and continuous line in the pictures) we have the same situation as in Figure 1, where the population is uniformly distributed across space with a constant unitary density, while in the second (the red and dotted line in the picture) the population has the same total size but is concentrated in the high productivity zones. In this second case two distinct motivations drive the planner: on one hand, she will tend to invest more in the more productive areas, but on the other, she is tempted to assign a reasonable enough per capita level of consumption in each region (again due to the Benthamite form of the utility functional). The total effect is depicted in the various pictures of Figure 3: the aggregate investment in more productive

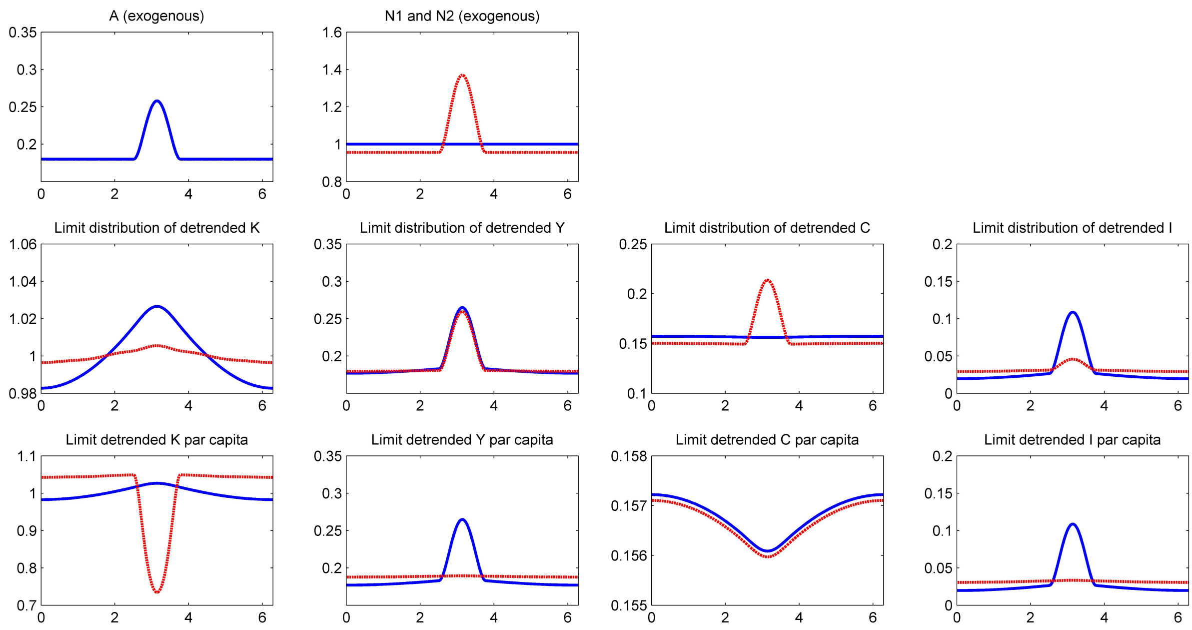

Figure 3. Core-periphery and population effect at work

areas for the second population profile remains relatively higher[13] but the effect is mitigated because aggregate consumption is higher in these areas as well. All in all the distribution of long-run detrended capital is much more uniform in the second case so that capital accumulates relatively more in less productive areas. For this reason the change in the population distribution translates in a sort of loss of efficiency of the system: as one can see (third picture of the third line of Figure 3), per-capita consumption in the new configuration is always smaller that in the original one at any location.

6. Conclusions

In this paper we introduce and study a general spatial model of economic growth. We are able to solve it analytically by using dynamic programming in infinite dimensions. This is made possible thanks to the use of the eigenfunctions of the linear Sturm-Liouville problem related to the consumption-free dynamics of the model. With respect to previous related contributions, our model is more general both for the possibility of studying heterogeneous spatial distributions of technology and for allowing for non-homogeneous spatially distributed population. The numerical simulation identifies the two key effects shaping the long-run configuration: core-periphery polarization and dilution.

[13]This outcome depends on the chosen distribution of the population, a bigger concentration of the population would of course accentuate the population effect.

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R. Boucekkine, Aix-Marseille University (Aix-Marseille School of Economics), CNRS and EHESS. 2 Rue de la Charité, 13002 Marseille, France.
E-mail address: raouf.boucekkine@univ-amu.fr

G. Fabbri, Aix-Marseille University (Aix-Marseille School of Economics), CNRS and EHESS. 2 Rue de la Charité, 13002 Marseille, France.
E-mail address: giorgio.fabbri@univ-amu.fr

S. Federico, Università degli Studi di Siena, Dipartimento di Economia Politica e Statistica. Piazza San Francesco 7-8, 53100 Siena , Italy.
E-mail address: salvatore.federico@unisi.it

F. Gozzi, Dipartimento di Economia e Finanza, Libera Università degli Studi Sociali *Guido Carli*, Roma
E-mail address: fgozzi@luiss.it