Another perspective on Borda's paradox*

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Abstract

This paper presents the conditions required for a profile in order to never exhibit either the strong or the strict Borda paradoxes under all weighted scoring rules in three-candidate elections. The main particularity of our paper is that all the conclusions are extracted from the differences of votes between candidates in pairwise majority elections. This way allows us to answer new questions and provide an organized knowledge of the conditions under which a given profile never shows one of the two paradoxes.

Keywords: Voting, Geometry, Borda's Paradox, Condorcet Pairwise Procedure, Weighted Scoring Rules.

JEL classification: D71, D72.

1 Introduction

Collective rationality has been a central property of social choice in economic theory. The eighteenth century saw the first thinking about various counterintuitive or paradoxical events that could occur when a group of voters aim to select a best candidate from a set of available candidates. Marquis de Condorcet (1785) and Borda (1781) have exhibited a difficulty concerning collective rationality through a given configuration of individual preferences that could create non coherent social outcomes using the well-known Plurality rule. Under this rule, each voter ranks the candidates in her order of preference and then we count the number of times each candidate is ranked first. Finally, the candidate who receives a relative majority (i.e., more votes than any of her competitors) is elected. Condorcet (1785) proposed a system for electing candidates who truly command pairwise majority support. In this system, a voter has the opportunity to rank all candidates from best to worst and once the individual ballots are submitted, the system considers all of the different pairwise majorities of two candidates against one another. The winner, also known as the Condorcet winner, is the candidate that wins all of the pairwise elections she is involved in. Borda (1781) also had serious doubts about the Plurality system, and therefore, defended a method which is today named after him. If a group must rank m candidates such that each individual has already ranked them, the Borda voting rule

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determines the winner as follows. In each individual ranking, Borda rule assigns $m-1$ to the top-ranked candidate, $m-2$ to the second-ranked, until the candidate ranked last who receives zero points. For each of the candidates, the points are summed across all the ballots, and the candidate with the most points is the winner.

One of the particularly troublesome observation made by Borda (1781) is related to the possibility for Plurality rule to elect the Condorcet loser, i.e., the candidate who is defeated by every other candidate in pairwise majority elections. Indeed, when the alternative ranked first by more voters than any other alternative is defeated by every other alternative in pairwise majority elections, then we refer to such an outcome as the *strong Borda paradox*. This defines the Borda's major concern in his discussion initiated with Condorcet about such issues. However, Plurality rule can exhibit a more problematic collective outcome since it can completely reverse the ranking of the pairwise majority elections, i.e., the loser by the pairwise majority rule is elected and, at the same time, the winner by the pairwise majority rule is ranked last by Plurality rule. This phenomenon is called the *strict Borda paradox* in social choice literature.

The current study focuses on the conditions under which the two forms of Borda's paradox might be observed under all weighted scoring rules. In general, under a weighted scoring rule, each candidate gets some points from each voter according to her position in the voter's preference, and the candidate with the highest aggregated score is elected. As a direct consequence of this definition, both Borda and Plurality rules are considered as specific examples of this category of voting rules. We begin with a description of the related literature and then we present the originality of the contributions presented in our paper.

Related literature

Many research papers have already analyzed the occurrence of the two forms of Borda's paradox. Indeed, we know from the literature that Borda's paradox is not limited to elections that use Plurality rule. More precisely, all weighted scoring rules other than Borda rule can exhibit the strict and the strong Borda paradoxes. Daunou (1803) showed that Borda rule cannot rank the Condorcet winner in last place in an election, which means that the strict Borda paradox cannot occur. In addition, both Gärdenfors (1973) and Smith (1973) proved that Borda rule is the only weighted scoring rule that has this property for sufficiently large number of voters. In other words, the strict Borda paradox can occur for all weighted scoring rules other than Borda rule. Fishburn and Gehrlein (1976) extend these results and show that Borda rule is the only weighted scoring rule that cannot select the Condorcet loser for sufficiently large number of voters. That is to say that Borda rule is the only weighted scoring rule that cannot exhibit the strong Borda paradox. Gehrlein and Lepelley (2010b) examined the probability that the two paradoxes will be observed with Plurality rule and Negative Plurality rule (another specific weighted scoring rule, which is defined later) under a well-known assumption about the voters' preferences. Diss and Gehrlein (2012) extend that analysis by considering all the class of weighted scoring rules and taking into account different assumptions about the individuals' preferences. Notice that Gehrlein and Lepelley (2010b) also analyzed the impact that degrees of mutual coherence among voters' preferences will have on the likelihood of observing the two forms of Borda's paradox under Plurality rule and Negative Plurality rule. Kamwa and Valognes (2017) use an analytical approach to identify the range of all weighted scoring rules that never exhibit the strong Borda paradox under some restrictions on the individuals' preferences. In addition, they provide the likelihood of the strong Borda paradox given these restrictions of preferences. Finally, it is worthwhile to mention that some empirical studies have been conducted in order to discuss if the two forms of Borda's paradox have occurred in real-life elections. The reader is refereed to Diss and Gehrlein (2012) and Gehrlein and Lepelley (2010a).

Our contributions

We extend the results of Saari and McIntee (2013) in order to answer questions about the required conditions for a profile[1] to never exhibit either the strong or the strict Borda paradoxes under all weighted scoring rules and any given number of voters in three-candidate elections. The new technique developed in Saari and McIntee (2013) identifies all profiles that satisfy any specified pairwise and weighted criteria, such as where Condorcet winner is ranked first by any weighted scoring rule. More exactly, all the conclusions are extracted from the differences of votes between candidates in pairwise majority elections.

By doing so, our results allow us to precise, for instance, if any weighted scoring rule can never exhibit the two forms of Borda's paradox when at least one candidate beats another by receiving over 65% of the vote, or where nobody receives more than 75% of the vote. The approach that we consider in this paper affords an opportunity to advance our understanding of which profiles create the strict and the strong Borda paradoxes. In other words, this paper provides an organized knowledge of the conditions for a profile to (never) show Borda's paradox. In sum, we have investigated a simple framework allowing us to answer a large variety of new questions. For instance, the results of this paper allow us to:

(i) Determine the minimum number of voters needed for a profile to show either the strong or the strict Borda paradoxes when the differences between candidates and the weighted scoring rule are already determined.

(ii) Give the differences between candidates in the pairwise election outcomes required for a profile to never exhibit one of the two paradoxes for a given weighted scoring rule and a fixed number of voters.

(iii) Describe what range of weighted scoring rules could possibly accompany a given number of voters and specified differences between candidates in the pairwise election outcomes in order for a profile to never exhibit one of the two paradoxes.

The rest of this paper is structured as follows. Section 2 presents the basic terminology, including the geometric tools needed to understand the next developments of this paper. Section 3 describes our results by giving the conditions that allow to never exhibit Borda's paradox. These results are described in Theorems (1-4) and Corollaries (1-4). Section 4 presents some comments and conclusions, and finally proofs are given in Section 5.

2 Preliminaries

2.1 Basic Framework

Throughout the paper, we will consider the framework of three-candidate elections that can be defined by a set of candidates, or alternatives, $\{A, B, C\}$ and a set of n voters $\{1, \ldots, n\}$. A linear order is a transitive, antisymmetric, and total relation. Each voter is assumed to have a linear order on the set of candidates from the most desirable candidate to the least

[1] A rigorous definition of this notion is given in Section 2.1.

desirable one. In addition, each voter is assumed to act according to her true preferences (i.e., to vote sincerely). The six possible strict rankings that voters might have are presented as follows:

No.	Ranking	No.	Ranking	No.	Ranking
1	$A \succ B \succ C$	2	$A \succ C \succ B$	3	$C \succ A \succ B$
4	$C \succ B \succ A$	5	$B \succ C \succ A$	6	$B \succ A \succ C$

(1)

The notation $A \succ B \succ C$ means that voters have candidate A as most preferred, candidate C as least preferred, and candidate B as middle-ranked. In this context, a profile is a sequence of n linear orders over $\{A, B, C\}$. Each profile can be defined by the vector $\tilde{n} = (n_1, n_2, \ldots, n_6)$, where in parentheses, we refer the number of voters n_i endowed with each of the six orders such that $\sum_{i=1}^{6} n_i = n$. For instance, n_1 is the number of voters endowed with the linear order $A \succ B \succ C$.

Given two candidates A and B, we shall write $A\mathbf{M}B$ to say that candidate A beats candidate B in a pairwise majority election. This occurs when a strict majority of voters prefer A to B. A given candidate is said to be a Condorcet winner if she wins in all of her pairwise elections. For instance, using our notations in (1), A is a Condorcet winner if $A\mathbf{M}B$ and $A\mathbf{M}C$, which is equivalent to $n_1+n_2+n_3 > n_4+n_5+n_6$ and $n_1+n_2+n_6 > n_3+n_4+n_5$, respectively. Conversely, a given candidate is said to be a Condorcet loser if she loses to everyone. For instance, C is a Condorcet loser if $A\mathbf{M}C$ and $B\mathbf{M}C$, which is equivalent to $n_1+n_2+n_6 > n_3+n_4+n_5$ and $n_1+n_5+n_6 > n_2+n_3+n_4$, respectively.

Without loss of generality, we assume throughout the paper that $A\mathbf{M}B$ and $B\mathbf{M}C$. With this assumption, if there is a strict transitive outcome of the paired comparisons, it has the form $A\mathbf{M}B\mathbf{M}C$. In other words, if the election has a Condorcet winner, it always is candidate A, and candidate C always is the Condorcet loser. In contrast, if there is a cyclical outcome, it always has the form $A\mathbf{M}B$, $B\mathbf{M}C$ and $C\mathbf{M}A$. This assumption does not affect the generality of our results since a name change converts any other situation into our framework.

Let XY denote the difference between X's and Y's majority vote in the pairwise comparison between X and Y.[2] Illustrating with a simple example with $n = 85$ voters distributed as follows: $n_1 = 25$, $n_2 = 15$, $n_3 = 20$, $n_4 = 5$, $n_5 = 10$ and $n_6 = 10$. It results that $AB = 60 - 25 = 35$, $BC = 45 - 40 = 5$ and $AC = 50 - 35 = 15$. Naturally, three immediate consequences follow from the XY values. First, the larger the XY value is, the better X does against Y. Second, our assumption of $A\mathbf{M}B$ and $B\mathbf{M}C$ is equivalent to $AB > 0$ and $BC > 0$, respectively. Third, the equation (2) is always satisfied.

$$XY \quad = \quad -YX. \tag{2}$$

Similarly to Saari and McIntee (2013), we will use a technical assumption, which is needed to separate the different cases of our results. This condition is defined as follows:

Definition 1 *A profile satisfies the strongly non-cyclic condition if*

$$AC \quad \geq \quad \min(AB, BC). \tag{3}$$

With our assumption of $AB > 0$ and $BC > 0$, it is obvious that if the condition (3) is satisfied, this leads to $AC > 0$. On the contrary, it is important to point out that if the condition (3) fails to hold because of weaker AC, our assumption of $AB > 0$ and $BC > 0$

[2] Saari and McIntee (2013) use the notation $P(X, Y)$. Instead, we use the notation XY since it allows us to reduce the length of our theorems and corollaries.

does not necessarily define a cycle. A simple example with $AB = 6$, $BC = 4$ and $AC = 2$ illustrates this fact.

In this paper, we are interested in the well-known weighted scoring rules, also called positional rules. Any weighted scoring rule in three-candidate elections can be defined by the vector $S = (1, s, 0)$ such that $0 \leq s \leq 1$. In other words, each of the n voters ranks the 3 candidates and we assign 1 point to the one ranked first, s points to the one ranked second, and 0 to the candidate ranked last. Predictably, the candidate that accumulates the most points, summed over all voters, wins the election. In this setting, the most commonly studied weighted scoring rules are Plurality rule with $s = 0$, Negative Plurality rule with $s = 1$ and Borda rule with $s = \frac{1}{2}$. Given two candidates A and B, we shall write $A\mathbf{S}B$ to say that candidate A beats candidate B when the weighted scoring rule S is used. As a consequence, A is the overall winner of the election if $A\mathbf{S}B$ and $A\mathbf{S}C$.

2.2 Geometry

We follow the Saari's methodology and some of his geometric arguments using the well-known triangle. In the current section, we only outline a summary of the very basic steps needed to understand the next developments of this paper. For a detailed description of Saari's voting geometry, the reader is referred to Saari (1989, 1995, 2010).[3] We particularly use the new way to analyze three-candidate relationships among pairwise and positional election outcomes that is developed in Saari and McIntee (2013). In the following, we present an overview of this methodology, and full details about this procedure are given in the original paper.

The geometric approach starts by representing three-candidate profiles in an equilateral triangle, where the name of each candidate is identified with a distinct vertex. Then, we draw a line between each vertex and the midpoint of the opposite segment (perpendicular bisector). What results is a partitioning of the triangle into six ranking regions. A point in the triangle defines a ranking according to its distances to the three vertices, where closer is better. We place n_i, the number of voters with the ith preference ranking, in each region as illustrated in Figure 1-a. For instance, any point in the small triangle with a n_1 value is closest to A, next closest to B, and farthest from C, so it has the $A \succ B \succ C$ strict linear order. It follows from this figure that, in a pairwise majority election between A and B, for instance, we just need to sum the numbers on each side of perpendicular bisector of the edge between A and B. In other words, the vertical line separates preferences where A is preferred to B (on the left) from the others where B is preferred to A (on the right). The analysis for the remaining two pairs is similar. This summing process is illustrated by the example in the left-hand side of Figure 1-b. This profile, which corresponds to our example in Section 2.1 with $n = 85$ voters, leads to $A\mathbf{M}B$ (60 votes for A and 25 for B), $B\mathbf{M}C$ (45 votes for B and 40 for C) and $A\mathbf{M}C$ (50 votes for B and 35 for C). Again, this is equivalent to $AB = 35$, $BC = 5$ and $AC = 15$.

The next step is to use this triangle to specify the outcome under a weighted voting rule $S = (1, s, 0)$. Using the triangle in Figure 1-a, the score of each candidate under Plurality rule is given by the sum of entries in the two regions sharing the corresponding vertex. For instance, the score of Plurality of A is $n_1 + n_2$. To put it in more general terms, the score of each candidate under the weighted scoring rule $S = (1, s, 0)$ is equal to her {Plurality score} plus {$s\times$ the number of voters who have her second ranked}. This leads to, respectively, $n_1 + n_2 + s(n_3 + n_6)$, $n_5 + n_6 + s(n_1 + n_4)$ and $n_3 + n_4 + s(n_2 + n_5)$, the scores of A, B and C. In our example described in the left-hand side of Figure 1-b, the scores

[3] See also Nurmi (2002).

of A, B and C are, respectively, $40 + 30s$, $20 + 30s$ and $25 + 25s$. With these scores, the Condorcet winner A is the winner under any weighted scoring rule S and the Condorcet loser C always is ranked second, while B always is ranked in the last position, with the exception of $s = 1$, where B and C are tied in the second position.

Using the XY values, Saari and McIntee (2013) present the following notions.

Definition 2 *For specified AB, BC and AC values, the essential profile is the profile with the smallest number of voters that have the specified XY values.*

Definition 3 *For a given number of voters n, a supporting profile is any profile that has the specified XY values of the essential profile.*

Saari and McIntee (2013) present all essential profiles (four of them) and the corresponding number of voters in each one. The results can be summarized as follows:[4]

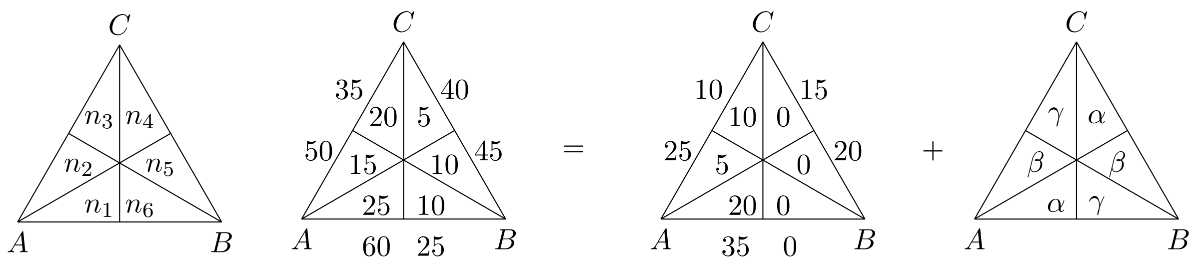
1. With the strongly non-cyclic condition, the essential profile has one of the Figure 2 a-c forms. The three possible cases are:
   - If AB is the largest value, then the essential profile has $n_4 = n_5 = n_6 = 0$ (Figure 2-a). The number of voters in this essential profile is AB.
   - If AC is the largest value, then the essential profile has $n_3 = n_4 = n_5 = 0$ (Figure 2-b). The number of voters in this essential profile is AC.
   - If BC is the largest value, then the essential profile has $n_2 = n_3 = n_4 = 0$ (Figure 2-c). The number of voters in this essential profile is BC.
2. If the strongly non-cyclic condition is not satisfied, the "cyclic" essential profile[5] has $n_2 = n_4 = n_6 = 0$ (Figure 2-d). The number of voters in this essential profile is $AB + BC - AC$.
3. In each essential profile, the number of voters endowed with the ith linear ranking $X \succ Y \succ Z$ is given by e_i such that:

$$e_i = \frac{1}{2}(XY + YZ). \tag{4}$$

The essential profile of the original $n = 85$ voters example is described in the middle of Figure 1-b with $n_4 = n_5 = n_6 = 0$ and the number of voters is equal to $AB = 35$. It follows from (4) that e_1, which represents the ranking $A \succ B \succ C$, is given by $e_1 = \frac{1}{2}(AB + BC) = \frac{1}{2}(35 + 5) = 20$. The same logic is used for $e_2 = \frac{1}{2}(AC + CB) = \frac{1}{2}(15 - 5) = 5$ and $e_3 = \frac{1}{2}(CA + AB) = \frac{1}{2}(-15 + 35) = 10$.

[4]Notice that these results are found under the standard assumption that $AB > 0$ and $BC > 0$. This is established without loss of generality.

[5]Recall that, if the strongly non-cyclic condition is not satisfied, this does not necessarily define a cycle. Although we use the name "cyclic" for this essential profile (to be consistent with Saari and McIntee (2013)), we need to consider $AC > 0$, in our framework of Borda's paradox, in order for C to be a Condorcet loser.



a. Profiles b. Example

Figure 1: Example when AB dominates

The decomposition approach is based on the fact that, to remove terms from the initial profile without affecting XY values, the method called *reversal pairs*[6] is the only way[7] as shown in Saari (1999, 2008). More precisely, to find the essential profile just remove as many reversal pairs as possible. Illustrating with the original example $(25, 15, 20, 5, 10, 10)$ with $n = 85$ voters. Removing pairs $(5, 0, 0, 5, 0, 0)$[8], $(0, 10, 0, 0, 10, 0)$, and $(0, 0, 10, 0, 0, 10)$ creates its essential profile $(20, 5, 10, 0, 0, 0)$ with 35 voters. Using an essential profile, it becomes possible to identify all possible profiles having the specified XY values (i.e., the supporting profiles) by only adding reversal pairs to the essential profile. With the original $n = 85$ voter example, 50 (85-35) voters must be added in appropriate ways in order to never affect the XY values. Naturally, the initial example presented in the left-hand side of Figure 1-b is a possibility by adding pairs $(5, 0, 0, 5, 0, 0)$, $(0, 10, 0, 0, 10, 0)$, and $(0, 0, 10, 0, 0, 10)$. Another possibility would be to add pairs $(1, 0, 0, 1, 0, 0)$, $(0, 2, 0, 0, 2, 0)$ and $(0, 0, 22, 0, 0, 22)$ leading to the supporting profile $(21, 7, 32, 1, 2, 22)$ or also adding pairs $(2, 0, 0, 2, 0, 0)$, $(0, 5, 0, 0, 5, 0)$ and $(0, 0, 18, 0, 0, 18)$ leading to the supporting profile $(22, 10, 28, 2, 5, 18)$. More generally, this can be done by adding pairs $(\alpha, 0, 0, \alpha, 0, 0)$, $(0, \beta, 0, 0, \beta, 0)$, and $(0, 0, \gamma, 0, 0, \gamma)$ to the essential profile such that $\alpha + \beta + \gamma = \frac{50}{2}$. In our example in the right-hand side of Figure 1-b, the decomposition method leads to $\alpha = 5$ and $\beta = \gamma = 10$.

To summarize, it follows from Saari and McIntee (2013) that, with the strongly non-cyclic condition and if XY is the maximum pairwise victory, the number of pairs added to the essential profile is given by $2(\alpha + \beta + \gamma)$ such that:[9]

$$\alpha + \beta + \gamma = \frac{1}{2}(n - XY) \tag{5}$$

This is true because, as mentioned above, the number of voters in an essential profile, when XY is the maximum pairwise victory, is XY. If the strongly non-cyclic condition is not satisfied, the number of pairs added to the essential profile is given by $2(\alpha + \beta + \gamma)$ and the value of $\alpha + \beta + \gamma$ becomes:

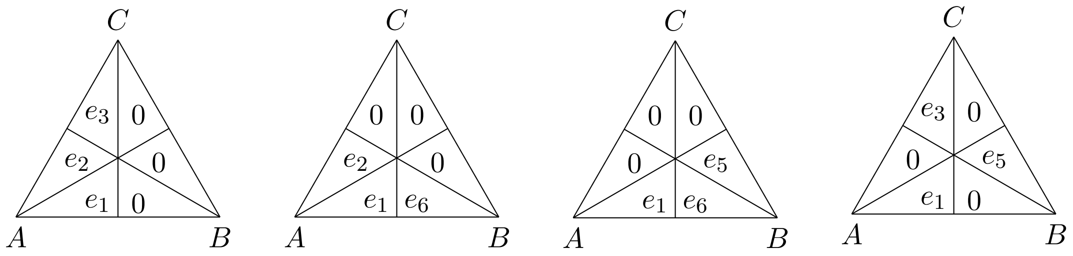
$$\alpha + \beta + \gamma = \frac{1}{2}[n - (AB + BC - AC)] \tag{6}$$

[6]For instance, the preference $A \succ B \succ C$ type has a reverse given by $C \succ B \succ A$. It is clear that, removing the same number of voters with reversal pairs does not change XY values.

[7]This is not true with more than three candidates.

[8]It means to remove 5 voters of each of both $A \succ B \succ C$ and $C \succ B \succ A$ preference types.

[9]Notice that, n and all XY values have the same parity. This is true because, for each pair of candidates X and Y, we have $n = \{X\text{'s vote} + Y\text{'s vote}\}$, which leads to $\{X\text{'s vote}\} = \frac{1}{2}(n + XY)$ since $XY = \{X\text{'s vote} - Y\text{'s vote}\}$. In order for $\{X\text{'s vote}\}$ to be an integer value, it follows that n and all XY values either are odd integers, or all are even integers. In other words, $\alpha + \beta + \gamma$ always is an integer value and, even more, it has the same parity as n and all XY values.



a. AB dominates b. AC c. BC d. Cyclic

Figure 2: The four essential profiles

It is important to point out that this approach allows Saari and McIntee (2013) to collect all profiles in equivalence classes by decomposing each profile into the part giving the essential profile - this part determines the XY values - and the part that affects only positional outcomes. Two profiles belong to the same equivalence class if they have in common the same specified XY values. It is clear that each equivalence class includes the class's essential profile as a component plus the other profiles, i.e., supporting profiles. As noted by Saari and McIntee (2013), this decomposition simplifies discovering and proving new conclusions. This approach will be used throughout this paper to tackle the strong and the strict Borda paradoxes.

3 Results

This section describes the restrictions on XY values that allow to never exhibit Borda's paradox under all weighted scoring rules $S = (1, s, 0)$ and any number of voters n. Proofs are in Section 5. We begin by providing our results for the strong Borda paradox.

3.1 Strong Borda Paradox

In our setting, the strong Borda paradox occurs if the Condorcet loser C is the winner of the weighted scoring rule S. In other words, $C\mathbf{S}A$ and $C\mathbf{S}B$. Theorems 1 and 2 specify what happens with the strong Borda paradox for all S. The particular cases of Plurality ($s = 0$) and Negative Plurality ($s = 1$) rules are described in Corollaries 1 and 2.

Theorem 1 *If the strongly non-cyclic condition is satisfied, the supporting profiles that never exhibit the strong Borda paradox are described as follows:*

1. *When AB is the largest pairwise victory, then three cases arise:*

   a. *If $s > \frac{1}{2}$ and $(3s-1)AB + 2(1-s)AC + sBC \geq (2s-1)n$*

   b. *If $s < \frac{1}{2}$, $(1-s)(AC-AB) + 2sBC < 0$ and $(1-s)AB + 2(1-s)AC + sBC \geq (1-2s)n$*

   c. *If $s < \frac{1}{2}$, $(1-s)(AC-AB) + 2sBC \geq 0$ and $(1-s)AC + sBC \geq \frac{1-2s}{3}(n-2+2F(U))$*

2. *When AC is the largest pairwise victory, then two cases arise:*

   a. *If $s > \frac{1}{2}$ and $(1-s)AB + 2sAC + sBC \geq (2s-1)n$*

b. *If* $s < \frac{1}{2}$ *and* $(1-s)AC + sBC \geq \frac{1-2s}{3}(n-2+2F(U))$

3. *When* BC *is the largest pairwise victory, then three cases arise:*

   a. *If* $s > \frac{1}{2}$, $2(1-s)AB + s(AC - BC) \geq 0$ *and* $(1-s)AB + 2sAC + sBC \geq (2s-1)n$

   b. *If* $s > \frac{1}{2}$, $2(1-s)AB + s(AC - BC) < 0$ *and* $(s-1)AB + sAC + 2sBC \geq (2s-1)n$

   c. *If* $s < \frac{1}{2}$ *and* $sAC + (1-s)BC \geq \frac{1-2s}{3}(n-2+2F(U))$

   Such that $U = \frac{(1-s)(AC-AB) + 2sBC}{2(1-2s)}$ *and* $F(U)$ *is its fractional part.*[10]

If the strongly non-cyclic condition is not satisfied, the requirements for the non existence of Borda's paradox are similar in form.

Theorem 2 *If the strongly non-cyclic condition is not satisfied, the supporting profiles that never exhibit the strong Borda paradox are described as follows:*

a. *If* $s > \frac{1}{2}$, $AB + (1-s)AC - sBC \geq 0$ *and* $(3s-1)AB + 2(1-s)AC + sBC \geq (2s-1)n$

b. *If* $s > \frac{1}{2}$, $AB + (1-s)AC - sBC < 0$ *and* $(3s-2)AB + (1-s)AC + 2sBC \geq (2s-1)n$

c. *If* $s < \frac{1}{2}$, $(1-s)AB - sAC - BC > 0$ *and* $(1-s)AB + 2sAC + (2-3s)BC \geq (1-2s)n$

d. *If* $s < \frac{1}{2}$, $(1-s)AB - sAC - BC \leq 0$ *and* $(1-s)BC + sAC \geq \frac{1-2s}{3}(n-2+2F(U))$

Such that $U = \frac{(s-1)AB + sAC + BC}{2(1-2s)}$ *and* $F(U)$ *is its fractional part.*

The immediate consequence is that these theorems allow to deduce the conditions under which the strong Borda paradox does not occur with Plurality ($s = 0$) and Negative Plurality ($s = 1$) rules. A listing of what can happen follows:

Corollary 1 *For Plurality rule, the supporting profiles under which the strong Borda paradox never occurs are described as follows:*

1. *If the strongly non-cyclic condition is satisfied:* ***a)*** *When* AB *dominates and* $AB + 2AC \geq n$. ***b)*** *When* AC *dominates and* $3AC \geq n-2$.[11] ***c)*** *When* BC *dominates and* $3BC \geq n-2$

2. *If the strongly non-cyclic condition is not satisfied:* ***a)*** *When* $AB > BC$ *and* $AB+2BC \geq n$. ***b)*** *When* $AB \leq BC$ *and* $3BC \geq n-2$.

[10]Recall that for all U in $\mathbb{Z}$, we have $\lfloor U \rfloor = U - F(U)$ such that $\lfloor U \rfloor$ stands for the greatest integer less than or equal to U. Notice that $F(U)$ is defined in the same way for positive and negative numbers. However, it is important to precise that for instance $\lfloor -1.3 \rfloor = -2$ leading to $F(-1.3) = 0.7$ and $\lfloor 1.3 \rfloor = 1$ leading to $F(1.3) = 0.3$.

[11]Notice that, in this case, $U = \frac{AC - AB}{2}$ and $F(U) = 0$ since AB and AC have the same parity (see discussion in footnote 9). The same remark can be used for some other parts of our corollaries.

Corollary 2 *For Negative Plurality rule, the supporting profiles under which the strong Borda paradox never occurs are described as follows:*

1. *If the strongly non-cyclic condition is satisfied:* ***a)*** *When AB dominates and $2AB + BC \geq n$.* ***b)*** *When AC dominates and $2AC + BC \geq n$.* ***c)*** *When BC dominates and $AC + 2BC \geq n$.*

2. *If the strongly non-cyclic condition is not satisfied:* ***a)*** *When $AB \geq BC$ and $2AB+BC \geq n$.* ***b)*** *When $AB < BC$ and $AB + 2BC \geq n$.*

We can deduce the following facts from Theorems 1 and 2 and Corollaries 1 and 2.

- The first lesson to be drawn from our results is that, for fixed values of XY and a given weighted scoring rule S, our results provide the minimum number of voters to obtain the strong Borda paradox. For instance, consider the XY values in Figure 1-b where $AB = 35$, $AC = 15$ and $BC = 5$. It follows that the strong Borda paradox never occurs under Plurality rule if $n \leq 65$, i.e., $\alpha + \beta + \gamma \leq (65 - 35)/2 = 15$. However, if $n > 65$, i.e., $\alpha + \beta + \gamma > 15$, supporting profiles can be constructed such that the strong Borda paradox occurs. In other words, for a profile to show the strong Borda paradox, the smallest value is $\alpha + \beta + \gamma = 16$ leading to $n = 67$. In this case, suppose that $\gamma = 16$ and $\alpha = \beta = 0$, which is equivalent to the profile $\tilde{n} = (20, 5, 10, 0, 0, 0) + (0, 0, 16, 0, 0, 16) = (20, 5, 26, 0, 0, 16)$. As a consequence, it is easy to verify that this profile exhibits the strong Borda paradox under Plurality rule. For Negative Plurality rule and the same XY values, the strong Borda paradox never occurs if $n \leq 75$, while some supporting profiles showing the strong Borda paradox can be found if $n > 75$, i.e., $\alpha + \beta + \gamma > (75 - 35)/2 = 20$. Suppose for example that $\alpha + \beta + \gamma = 21$ leading to $n = 77$, which is, in this case, the smallest number of voters for a profile to show the paradox. Let assume that $\beta = 21$ and $\alpha = \gamma = 0$, which is equivalent to $\tilde{n} = (20, 5, 10, 0, 0, 0) + (0, 21, 0, 0, 21, 0) = (20, 26, 10, 0, 21, 0)$. Clearly, this profile gives rise to the strong Borda paradox under Negative Plurality rule.

- The second lesson to be learned is that, for a given weighted scoring rule S and a fixed number of voters n, our results provide the conditions over the XY values required for a supporting profile to never exhibit the strong Borda paradox. For instance, with Plurality rule and $n = 85$ (the same number of voters as Figure 1-b), we definitely are assured to only have supporting profiles that never exhibit the strong Borda paradox when these profiles verify the strongly non-cyclic condition with AB dominating and $AB + 2AC \geq 85$. In other words, if $AC = 15$ and $BC = 5$ for instance, then all supporting profiles with $AB \geq 55$ never show the strong Borda paradox. If $AB < 55$, some supporting profiles exhibiting the paradox can be found. For instance, the profiles $\tilde{n} = (20, 5, 35, 0, 0, 25)$ and $\tilde{n} = (23, 7, 30, 3, 2, 20)$ with $AB = 35$.

- Another lesson learned from this paper is that our results provide all tools needed to describe what weighted scoring rule S could possibly accompany specified XY values and n in order for a supporting profile to never exhibit the strong Borda paradox. Let us consider a simple example with $AB = 4$, $AC = BC = 2$ and $n = 10$ (i.e., the strongly non-cyclic condition is satisfied and AB dominating). Then it is easy to show that all weighted scoring rules S with $\frac{1}{2} < s \leq 1$ never exhibit the strong Borda paradox. This is true because the part 1-a in Theorem 1 leads to $s > \frac{1}{2}$ and $s \leq 1$. In addition, with the same example, all weighted scoring rules in the range $\frac{1}{7} \leq s < \frac{1}{2}$ never show the strong Borda paradox. More precisely, the part 1-b in Theorem 1 leads to $\frac{1}{7} \leq s < \frac{1}{3}$ while the part 1-c gives rise to $\frac{1}{3} \leq s < \frac{1}{2}$. Let us explain more precisely the latter case. First, the condition $(1-s)(AC-AB)+2sBC \geq 0$ in 1-c of Theorem 1 leads to $s \geq \frac{1}{3}$. In addition, in our example, the value of $F(U)$ in the condition $(1-s)AC+sBC \geq \frac{1-2s}{3}(n-2+2F(U))$ does not matter. Indeed, we know that $0 \leq F(U) \lesssim 1$. In addition, if $F(U)=0$, the condition $(1-s)AC+sBC \geq \frac{1-2s}{3}(n-2+2F(U))$ becomes $s \geq \frac{1}{8}$. Moreover, if $F(U) \simeq 1$, the condition $(1-s)AC+sBC \geq \frac{1-2s}{3}(n-2+2F(U))$ becomes $s \gtrsim \frac{1}{5}$. Both $s \geq \frac{1}{8}$ and $s \gtrsim \frac{1}{5}$ are relaxed in comparison with the first requirement $s \geq \frac{1}{3}$.

Notice that, if we consider our example in Figure 1-b where $n=85$, $AB=35$, $AC=15$ and $BC=5$, we find that all weighted scoring rules in the ranges $\frac{1}{2} < s \leq \frac{8}{9}$ and $\frac{2}{11} \leq s < \frac{1}{2}$ never show the strong Borda paradox.

- Our last remark deals with the comparison of all weighted voting rules S and how the conditions change when s varies. In Theorems 1 and 2, it is easy to see that when $s \to \frac{1}{2}$, then the right-hand sides $(2s-1)n$ for $s > \frac{1}{2}$ and either $(1-2s)n$ or $\frac{1-2s}{3}(n-2+2F(U))$ for $s < \frac{1}{2}$ tend to zero. By avoiding a lower bound with n, the conditions become significantly more relaxed than those required for the other weighted scoring rules. For instance, the right-hand side of each condition of Corollary 1 for Plurality rule and Corollary 2 for Negative Plurality rule involve the n value. In other words, the region for which the strong Borda paradox does not occur is much more important when $s \to \frac{1}{2}$. This result is not new and is consistent with the existing literature dealing with the probability of the strong Borda paradox under various scenarios (e.g., Diss and Gehrlein, 2012; Gehrlein and Lepelley, 2010b). However, one of the particularity of our paper is that this result is general and it does not depend on the restrictions on the individuals' preferences.

3.2 Strict Borda Paradox

In our setting, the strict Borda paradox occurs if the Condorcet loser C is ranked first and the Condorcet winner A is ranked last by the weighted scoring rule $S=(1,s,0)$. In other words, the used weighted scoring rule would completely reverse the pairwise ranking on the candidates. Theorems 3 and 4 below generalize the restrictions on XY under which the strict Borda paradox occurs for all rules S and any number of voters n. The particular cases of Plurality ($s=0$) and Negative Plurality ($s=1$) rules are described in Corollaries 3 and 4.

Theorem 3 *If the strongly non-cyclic condition is satisfied, the supporting profiles that never exhibit the strict Borda paradox are described as follows:*

1. *When AB is the largest pairwise victory, then three cases arise:*

a. *If* $s > \frac{1}{2}$, $sAB + (1-s)AC \geq \frac{2s-1}{3}(n - 4 + 4F(U))$

b. *If* $s < \frac{1}{2}$, $(1-s)(AC-AB)+2sBC \geq 0$ *and* $sBC+(1-s)AC \geq \frac{1-2s}{3}(n-4+4F(V))$

c. *If* $s < \frac{1}{2}$, $(1-s)(AC-AB)+2sBC < 0$ *and* $2(1-s)AB-sBC+(1-s)AC \geq (1-2s)n$

2. *When* AC *is the largest pairwise victory, then two cases arise:*

a. *If* $s > \frac{1}{2}$ *and* $(1-s)AB + sAC \geq \frac{2s-1}{3}(n - 4 + 4F(U))$

b. *If* $s < \frac{1}{2}$ *and* $sBC + (1-s)AC \geq \frac{1-2s}{3}(n - 4 + 4F(V))$

3. *When* BC *is the largest pairwise victory, then three cases arise:*

a. *If* $s > \frac{1}{2}$, $2(1-s)AB+s(AC-BC) \geq 0$ *and* $(1-s)AB+sAC \geq \frac{2s-1}{3}(n-4+4F(U))$

b. *If* $s > \frac{1}{2}$, $2(1-s)AB + s(AC-BC) < 0$ *and* $(s-1)AB + 2sBC + sAC \geq (2s-1)n$

c. *If* $s < \frac{1}{2}$ *and* $sAC + (1-s)BC \geq \frac{1-2s}{3}(n - 4 + 4F(V))$

Where $U = \frac{AB+(1-s)AC-sBC}{2(2s-1)}$, $V = \frac{(1-s)(AC-AB)+2sBC}{2(1-2s)}$ *and* $F(U)$ *and* $F(V)$ *are the fractional parts of* U *and* V*, respectively.*

If the strongly non-cyclic condition is not satisfied, the requirements for the non existence of the strict Borda paradox are described below.

Theorem 4 *If the strongly non-cyclic condition is not satisfied, the supporting profiles that never exhibit the strict Borda paradox are described as follows:*

a. *If* $s > \frac{1}{2}$, $AB + AC - s(BC + AC) \geq 0$ *and* $sAB + (1-s)AC \geq \frac{2s-1}{3}(n - 4 + 4F(U))$

b. *If* $s > \frac{1}{2}$, $AB + AC - s(BC + AC) < 0$ *and* $(3s-2)AB + (1-s)AC + 2sBC \geq (2s-1)n$

c. *If* $s < \frac{1}{2}$, $BC - AB + s(AB + AC) < 0$ *and* $2(1-s)AB + (1-3s)BC + sAC \geq (1-2s)n$

d. *If* $s < \frac{1}{2}$, $BC - AB + s(AB + AC) \geq 0$ *and* $(1-s)BC + sAC \geq \frac{1-2s}{3}(n - 4 + 4F(U))$

Where $U = \frac{AB+(1-s)AC-sBC}{2(2s-1)}$, $V = \frac{(1-s)(AC-AB)+2sBC}{2(1-2s)}$ *and* $F(U)$ *and* $F(V)$ *are the fractional parts of* U *and* V*, respectively.*

The conditions under which the strict Borda paradox never occurs with the well-known Plurality ($s = 0$) and Negative Plurality ($s = 1$) rules are consequently described in the next corollaries.

Corollary 3 *For Plurality rule, the supporting profiles under which the strict Borda paradox never occurs are described as follows:*

1. *If the strongly non-cyclic condition is satisfied:* ***a)*** *When* AB *dominates and* $2AB + AC \geq n$*.* ***b)*** *When* AC *dominates and* $3AC \geq n - 4$*.* ***c)*** *When* BC *dominates and* $3BC \geq n - 4$*.*

2. *If the strongly non-cyclic condition is not satisfied:* ***a)*** *When $AB > BC$ and $2AB+BC \geq n$.* ***b)*** *When $AB \leq BC$ and $3BC \geq n-4$.*

Corollary 4 *For Negative Plurality rule, the supporting profiles under which the strict Borda paradox never occurs are described as follows:*

1. *If the strongly non-cyclic condition is satisfied:* ***a)*** *When AB dominates and $3AB \geq n-4$.* ***b)*** *When AC dominates and $3AC \geq n-4$.* ***c)*** *When BC dominates and $2BC+AC \geq n$.*

2. *If the strongly non-cyclic condition is not satisfied:* ***a)****When $AB \geq BC$ and $3AB \geq n-4$.* ***b)*** *When $AB < BC$ and $AB + 2BC \geq n$.*

Similarly to the previous section, we can deduce some interesting facts from Theorems 3 and 4 and Corollaries 3 and 4.

- For fixed values of XY and a given weighted scoring rule S, it is possible to deduce the minimum number of voters to ensure that a supporting profile exhibiting a strict Borda paradox can be constructed. It is clear that there exist supporting profiles for which the strict Borda paradox occurs if the inequalities are reversed in each case. For instance, let us again consider the XY values in Figure 1-b. We see from our results that the strict Borda paradox never occurs under Plurality rule if $n \leq 85$, i.e., $\alpha + \beta + \gamma \leq (85-35)/2 = 25$. Hence, if $n > 85$, i.e., $\alpha + \beta + \gamma > 25$, supporting profiles can be constructed such that the strict Borda paradox occurs. In other words, the minimum number of reversal pairs $\alpha + \beta + \gamma$ for which problems occur is 26. This leads to $n = 87$, the minimum number of voters for a profile to exhibit the strict Borda paradox for our XY values. An instance of this paradox is $\gamma = 26$ and $\alpha = \beta = 0$, which is equivalent to the profile $\tilde{n} = (20,5,10,0,0,0) + (0,0,26,0,0,26) = (20,5,36,0,0,26)$. It is easy to verify that the pairwise ranking on the candidates ($A\mathbf{M}B\mathbf{M}C$ and $A\mathbf{M}C$) is completely reversed by the Plurality rule. For Negative Plurality rule and the same XY values, the strict Borda paradox never occurs if $n \leq 109$, while some supporting profiles can be found if $n > 109$, i.e., $\alpha+\beta+\gamma > (109-35)/2 = 37$. Suppose for example that $\alpha + \beta + \gamma = 38$, i.e., $n = 111$. Let us assume that $\alpha = 16$, $\beta = 22$ and $\gamma = 0$, which is equivalent to $\tilde{n} = (20,5,10,0,0,0) + (16,22,0,16,22,0) = (36,27,10,16,22,0)$. This profile clearly gives rise to the strict Borda paradox under Negative Plurality rule and $n = 111$ is the minimum number of voters for a supporting profile to exhibit this paradox.

- For a given weighted scoring rule S and a fixed number of voters n, our results give the constraints over the XY values required for a profile to never show the strict Borda paradox. An instance can be given with Plurality rule and $n = 85$. If AB dominates with the strongly non-cyclic condition verified and $2AB+AC \geq 85$, then no supporting profiles can be found such that the strict Borda paradox occurs. Suppose for instance that $AC = 15$ and $BC = 5$. In this case, all supporting profiles with $AB \geq 35$ never show the strict Borda paradox. If $AB < 35$, some supporting profiles exhibiting the paradox can be constructed. For instance, the supporting profiles $\tilde{n} = (19,5,35,0,0,26)$ and $\tilde{n} = (19,6,34,0,1,25)$ with $AB = 33$.

- For specified XY values and a given number of voters n, we can deduce from our results all weighted scoring rules that never exhibit the strict Borda paradox. Let us consider the same values as the example considered in Section 3.1 where $AB = 4$, $AC = BC = 2$ and $n = 10$. This means that the strongly non-cyclic condition is satisfied and AB dominates. Hence all weighted scoring rules S with $\frac{1}{2} < s \leq 1$ never exhibit the strong Borda paradox. More precisely, the part 1-a in Theorem 3 leads to $s > \frac{1}{2}$ and either $s \leq 2$ with $F(U) = 0$ or $s \lesssim \frac{8}{7}$ with $F(U) \simeq 1$. In addition, with the same example, all weighted scoring rules in the range $0 \leq s < \frac{1}{2}$ never show the strict Borda paradox.

With the example described in Figure 1-b where $n = 85$, $AB = 35$, $AC = 15$ and $BC = 5$, we find that all weighted scoring rules $s \neq \frac{1}{2}$ never show the strict Borda paradox.

- The previous remark allows us to point out that the range of weighted scoring rules that never show the strict Borda paradox are much larger that those for the strong Borda paradox. In other words, the conditions of observing a strict Borda paradox that are listed in Theorems 3 and 4 are significantly more relaxed than the corresponding conditions for observing a strong Borda paradox in Theorems 1 and 2. For instance, as shown in the previous remark, with the example with $n = 85$, $AB = 35$, $AC = 15$ and $BC = 5$, we find that all weighted scoring rules $s \neq \frac{1}{2}$ never show the strict Borda paradox. However, at the same time, we have found in Section 3.1 that all weighted scoring rules in the ranges $\frac{1}{2} < s \leq \frac{8}{9}$ and $\frac{2}{11} \leq s < \frac{1}{2}$ never show the strong Borda paradox. The same remark is valid for the example with $n = 10$ where $AB = 4$ and $AC = BC = 2$. This result is also consistent with the literature on the probability of Borda's paradox (e.g., Diss and Gehrlein, 2012; Gehrlein and Lepelley, 2010b).

- Similarly to our remark for the strong Borda paradox, it is easy to see from Theorems 3 and 4 that, when $s \to \frac{1}{2}$, the right-hand sides $(2s-1)n$ or $\frac{2s-1}{3}(n-4+4F(U))$ for $s > \frac{1}{2}$ and either $(1-2s)n$ or $\frac{1-2s}{3}(n-4+4F(U))$ for $s < \frac{1}{2}$ tend to zero. In other words, the conditions to never exhibit the strict Borda paradox are significantly more relaxed with $s \to \frac{1}{2}$ in comparison with the other weighted scoring rules such as Plurality and Negative Plurality rules. This result is consistent with the existing literature dealing with the probability of the strict Borda paradox under various scenarios. Again, our result is general since it does not depend on the condition made on the voters' preferences.

4 Concluding remarks

We have used a new technique developed in Saari and McIntee (2013) to answer questions about the required conditions for the non-existence of either the strong or the strict Borda paradoxes under all weighted scoring rules $S = (1, s, 0)$ in three-candidate elections. This paper is reporting completely new results allowing us, for example, to (i) give the conditions over the XY values required for a supporting profile to never exhibit the strong and the strict Borda paradoxes for a given weighted scoring rule S and a fixed number of voters n. (ii) determine the minimum number of voters n needed for a supporting profile to show one of the two paradoxes when the XY values and the weighted scoring rule S are already determined. (iii) describe what range of weighted scoring rule S could possibly accompany specified XY and n values in order for a supporting profile to never exhibit one of the two paradoxes.

In other reports, we expect to adapt this technique in order to deal with other interesting questions. Our next objective will be to investigate the impact of an increasing degree of mutual coherence among the preferences of voters on some voting paradoxes for which many open questions still remain unanswered. For more details about metrics of group mutual coherence, the reader is referred to the book by Gehrlein and Lepelley (2010a) or their recent article in Gehrlein and Lepelley (2016).

5 Proofs

Theorems 1 and 3 are proved for the strongly non-cyclic case where AB has the largest victory. The other cases of these theorems as well as the case where the strongly non-cyclic condition is not satisfied (Theorems 2 and 4) are proved in the same manner. Proofs are described to let the reader understand what steps were taken in order to provide our results. Complete proofs are available upon request from the authors.

Proof of Theorem 1

Recall that, in our setting, the strong Borda paradox occurs if the Condorcet loser C is the winner of the weighted scoring rule defined by the vector $S = (1, s, 0)$, i.e., $C\mathbf{S}A$ and $C\mathbf{S}B$. Suppose AB is the largest pairwise victory, which means that our framework corresponds to the Figure 2.a. In other words, any profile $\tilde{n} = (n_1, n_2, \ldots, n_6)$ can be written as $\tilde{n} = (e_1 + \alpha, e_2 + \beta, e_3 + \gamma, \alpha, \beta, \gamma)$. By using the weighted scoring rule S, the scores of A, B and C are, respectively,

$$e_1 + e_2 + \alpha + \beta + s(e_3 + 2\gamma), \qquad \beta + \gamma + s(e_1 + 2\alpha), \qquad e_3 + \alpha + \gamma + s(e_2 + 2\beta).$$

It follows from these scores that $C\mathbf{S}A$ and $C\mathbf{S}B$ are equivalent to, respectively,

$$e_3 - e_1 - e_2 + s(e_2 - e_3) > (1 - 2s)\beta + (2s - 1)\gamma, \tag{7}$$
$$e_3 + s(e_2 - e_1) > (1 - 2s)\beta + (2s - 1)\alpha. \tag{8}$$

The $(1 - 2s)$ and $(2s - 1)$ terms on the right-hand side of each inequality differ in sign depending on whether $s > \frac{1}{2}$ or $s < \frac{1}{2}$. Then, we must consider two possible cases:

- **Case 1**: $s > \frac{1}{2}$. In this case, $(2s - 1) > 0$ and the inequalities (7-8) become

  $$\frac{e_3 - e_1 - e_2 + s(e_2 - e_3)}{2s - 1} > \gamma - \beta, \qquad \text{and} \qquad \frac{e_3 + s(e_2 - e_1)}{2s - 1} > \alpha - \beta. \tag{9}$$

  Or equivalently,

  $$\gamma < \frac{e_3 - e_1 - e_2 + s(e_2 - e_3)}{2s - 1} + \beta, \qquad \text{and} \qquad \alpha < \frac{e_3 + s(e_2 - e_1)}{2s - 1} + \beta. \tag{10}$$

  Knowing that α, β and γ all are positive integers, to ensure that a profile can be constructed, it follows that the minimal conditions for such a profile are where

$$\alpha_{min} = 0, \tag{11}$$

and

$$\gamma_{min} = 0, \tag{12}$$

and

$$\frac{e_3 - e_1 - e_2 + s(e_2 - e_3)}{2s-1} + \beta > 0, \tag{13}$$

and

$$\frac{e_3 + s(e_2 - e_1)}{2s-1} + \beta > 0. \tag{14}$$

The inequalities (13-14) are equivalent to, respectively,

$$\beta > \frac{-e_3 + e_1 + e_2 - s(e_2 - e_3)}{2s-1}, \quad \text{and} \quad \beta > \frac{-e_3 - s(e_2 - e_1)}{2s-1}. \tag{15}$$

It can easily be shown that the inequality in the right-hand side of (15) is relaxed since $\frac{-e_3 + e_1 + e_2 - s(e_2 - e_3)}{2s-1} > 0$ and $\frac{-e_3 + e_1 + e_2 - s(e_2 - e_3)}{2s-1} > \frac{-e_3 - s(e_2 - e_1)}{2s-1}$. As a consequence, the minimal conditions in (11-14) become:

$$\alpha_{min} = 0, \tag{16}$$

and

$$\gamma_{min} = 0, \tag{17}$$

and

$$\beta_{min} = \left\lfloor \frac{-e_3 + e_1 + e_2 - s(e_2 - e_3)}{2s-1} \right\rfloor + 1. \tag{18}$$

$\lfloor U \rfloor$ stands for the integer part of the number U. In all cases, the values of α, β and γ must satisfy $\alpha \geq \alpha_{min}$, $\beta \geq \beta_{min}$ and $\gamma \geq \gamma_{min}$. In addition, we know from (5) that $\alpha + \beta + \gamma = q = \frac{1}{2}(n - AB) = \frac{1}{2}(n - e_1 - e_2 - e_3)$. Then, the inequalities (16-18) lead to

$$\left\lfloor \frac{-e_3 + e_1 + e_2 - s(e_2 - e_3)}{2s-1} \right\rfloor + 1 \quad \leq \quad \frac{1}{2}(n - e_1 - e_2 - e_3). \tag{19}$$

Notice that $\lfloor U_1 \rfloor \leq U_2 \iff U_1 < U_2 + 1$ if U_2 is an integer. Taking into account the fact that $\frac{1}{2}(n - e_1 - e_2 - e_3) = \frac{1}{2}(n - AB)$ is an integer value since n and AB have the same parity (see discussion in footnote 9), the inequality (19) is equivalent to

$$\frac{-e_3 + e_1 + e_2 - s(e_2 - e_3)}{2s-1} \quad < \quad \frac{1}{2}(n - e_1 - e_2 - e_3). \tag{20}$$

Collecting terms using (4) leads to the condition (21), which defines the minimal requirement to guarantee the occurrence of the strong Borda paradox.

$$(3s-1)AB + 2(1-s)AC + sBC \quad < \quad (2s-1)n. \tag{21}$$

In other words, if the inequality (21) is reversed, then there is no profile for which the strong Borda paradox occurs.

- **Case 2**: $s < \frac{1}{2}$. In this case, $(2s-1) < 0$ and the inequalities (7-8) become

$$\frac{e_3 - e_1 - e_2 + s(e_2 - e_3)}{1-2s} > \beta - \gamma, \quad \text{and} \quad \frac{e_3 + s(e_2 - e_1)}{1-2s} > \beta - \alpha. \tag{22}$$

Or equivalently,

$$\beta < \gamma + \frac{e_3 - e_1 - e_2 + s(e_2 - e_3)}{1-2s}, \quad \text{and} \quad \beta < \alpha + \frac{e_3 + s(e_2 - e_1)}{1-2s}. \tag{23}$$

Knowing that α, β and γ all are positive integers, to ensure that a profile can be constructed, it follows that minimal conditions for such a profile are where

$$\beta_{min} = 0, \tag{24}$$

$$\text{and} \qquad \gamma > \frac{-e_3 + e_1 + e_2 - s(e_2 - e_3)}{1 - 2s}, \tag{25}$$

$$\text{and} \qquad \alpha > \frac{-e_3 - s(e_2 - e_1)}{1 - 2s}. \tag{26}$$

Notice that $\frac{-e_3 + e_1 + e_2 - s(e_2 - e_3)}{1 - 2s}$ always is positive. In other words,

$$\gamma_{min} = \left\lfloor \frac{-e_3 + e_1 + e_2 - s(e_2 - e_3)}{1 - 2s} \right\rfloor + 1. \tag{27}$$

However, $\frac{-e_3 - s(e_2 - e_1)}{1 - 2s}$, which is equivalent to $\frac{(1-s)(AC - AB) + 2sBC}{2(1 - 2s)}$, can be either positive or negative and depends on the sign of $(1 - s)(AC - AB) + 2sBC$. In other words, two cases arise for α_{min}.

- If $(1 - s)(AC - AB) + 2sBC < 0$. In this case,

  $$\alpha_{min} = 0. \tag{28}$$

  Using (24), and (27-28) and taking into account that the values of α, β and γ must satisfy $\alpha \geq \alpha_{min}$, $\beta \geq \beta_{min}$, $\gamma \geq \gamma_{min}$ and the equality $\alpha + \beta + \gamma = q = \frac{1}{2}(n - AB) = \frac{1}{2}(n - e_1 - e_2 - e_3)$, it follows that

  $$\left\lfloor \frac{e_1 + e_2 - e_3 - s(e_2 - e_3)}{1 - 2s} \right\rfloor + 1 \quad \leq \quad \frac{1}{2}(n - e_1 - e_2 - e_3). \tag{29}$$

  Using again $\lfloor U_1 \rfloor \leq U_2 \iff U_1 < U_2 + 1$ and collecting terms lead to

  $$(1 - s)AB + 2(1 - s)AC + sBC \quad < (1 - 2s)n. \tag{30}$$

  Finally, if this inequality is reversed, then there is no profile for which the strong Borda paradox occurs.

- If $(1 - s)(AC - AB) + 2sBC \geq 0$. In this case,

  $$\alpha_{min} = \left\lfloor \frac{-e_3 - s(e_2 - e_1)}{1 - 2s} \right\rfloor + 1. \tag{31}$$

  Using (24), (27) and (31) and the fact that the values of α, β and γ must satisfy $\alpha \geq \alpha_{min}$, $\beta \geq \beta_{min}$, $\gamma \geq \gamma_{min}$ and the equality $\alpha + \beta + \gamma = q = \frac{1}{2}(n - AB) = \frac{1}{2}(n - e_1 - e_2 - e_3)$, it follows

  $$\left\lfloor \frac{e_1 + e_2 - e_3 - s(e_2 - e_3)}{1 - 2s} \right\rfloor + \left\lfloor \frac{-e_3 - s(e_2 - e_1)}{1 - 2s} \right\rfloor + 2 \leq \frac{1}{2}(n - e_1 - e_2 - e_3). \tag{32}$$

  Or equivalently,

  $$\left\lfloor \frac{e_1 + e_2 - e_3 - s(e_2 - e_3)}{1 - 2s} \right\rfloor \leq \frac{1}{2}(n - e_1 - e_2 - e_3) - \left\lfloor \frac{-e_3 - s(e_2 - e_1)}{1 - 2s} \right\rfloor - 2. \tag{33}$$

Using again $\lfloor U_1 \rfloor \le U_2 \iff U_1 < U_2 + 1$, the inequality (33) is equivalent to

$$\frac{e_1 + e_2 - e_3 - s(e_2 - e_3)}{1 - 2s} < \frac{1}{2}(n - e_1 - e_2 - e_3) - \left\lfloor \frac{-e_3 - s(e_2 - e_1)}{1 - 2s} \right\rfloor - 1. \quad (34)$$

Notice that $\lfloor U \rfloor = U - F(U)$, such that $F(U)$ is the fractional part of U. Collecting terms of (34) leads to

$$(1-s)AC + sBC \quad < \quad \frac{1-2s}{3}(n - 2 + 2F(U)). \quad (35)$$

Such that $U = \dfrac{-e_3 - s(e_2 - e_1)}{1 - 2s} = \dfrac{(1-s)(AC - AB) + 2sBC}{2(1-2s)}$. In the same way, if the inequality (35) is reversed, then there is no profile for which the strong Borda paradox occurs. *Q.E.D.*

Proof of Theorem 3

Recall that, in our setting, the strict Borda paradox occurs if the Condorcet loser C is ranked first and the Condorcet winner A is ranked last by the weighted scoring rule $S = (1, s, 0)$. This is equivalent to $C\mathbf{S}B$ and $B\mathbf{S}A$. Suppose AB is the largest pairwise victory, which again refers to the framework of Figure 2.a. In other words, any profile $\tilde{n} = (n_1, n_2, \ldots, n_6)$ can be written as $\tilde{n} = (e_1 + \alpha, e_2 + \beta, e_3 + \gamma, \alpha, \beta, \gamma)$. Using the weighted scoring rule S, the scores of A, B and C are, respectively,

$$e_1 + e_2 + \alpha + \beta + s(e_3 + 2\gamma), \qquad \beta + \gamma + s(e_1 + 2\alpha), \qquad e_3 + \alpha + \gamma + s(e_2 + 2\beta).$$

It follows that $C\mathbf{S}B$ and $B\mathbf{S}A$ are equivalent to, respectively,

$$e_3 + s(e_2 - e_1) > (1 - 2s)\beta + (2s - 1)\alpha, \quad (36)$$
$$-(e_1 + e_2) + s(e_1 - e_3) > (1 - 2s)\alpha + (2s - 1)\gamma. \quad (37)$$

The $(1 - 2s)$ and $(2s - 1)$ terms on the right-hand side of each inequality differ in sign depending on whether $s > \frac{1}{2}$ or $s < \frac{1}{2}$. Then, we must consider two possible cases:

- **Case 1**: $s > \frac{1}{2}$. In this case, $(2s - 1) > 0$ and the inequalities (36-37) become

$$\frac{e_3 + s(e_2 - e_1)}{2s - 1} > \alpha - \beta, \qquad \text{and} \qquad \frac{-(e_1 + e_2) + s(e_1 - e_3)}{2s - 1} > \gamma - \alpha. \quad (38)$$

  Or equivalently,

$$\beta > \alpha + \frac{-e_3 + s(e_1 - e_2)}{2s - 1}, \qquad \text{and} \qquad \alpha > \gamma + \frac{e_1 + e_2 + s(e_3 - e_1)}{2s - 1}. \quad (39)$$

  Knowing that α, β and γ all are positive integers, to ensure that a profile can be constructed, it follows that the minimal conditions for such a profile are where

$$\gamma_{min} = 0, \quad (40)$$
$$\text{and} \qquad \alpha_{min} = \left\lfloor \frac{e_1 + e_2 + s(e_3 - e_1)}{2s - 1} \right\rfloor + 1, \quad (41)$$
$$\text{and} \qquad \beta_{min} = \alpha_{min} + \left\lfloor \frac{-e_3 + s(e_1 - e_2)}{2s - 1} \right\rfloor + 1. \quad (42)$$

This is true because $e_1 + e_2 + s(e_3 - e_1) > 0$ and $\alpha_{min} + \dfrac{-e_3 + s(e_1 - e_2)}{2s - 1} > 0$. $\lfloor U \rfloor$ stands for the integer part of the number U. In all cases, the values of α, β and γ must satisfy $\alpha \geq \alpha_{min}$, $\beta \geq \beta_{min}$ and $\gamma \geq \gamma_{min}$. In addition, we know from (5) that $\alpha + \beta + \gamma = q = \frac{1}{2}(n - AB) = \frac{1}{2}(n - e_1 - e_2 - e_3)$. Then, the equations (40-42) lead to

$$2\left\lfloor \frac{e_1 + e_2 + s(e_3 - e_1)}{2s - 1} \right\rfloor + \left\lfloor \frac{-e_3 + s(e_1 - e_2)}{2s - 1} \right\rfloor + 3 \quad \leq \quad \frac{1}{2}(n - e_1 - e_2 - e_3). \tag{43}$$

Or equivalently,

$$\left\lfloor \frac{-e_3 + s(e_1 - e_2)}{2s - 1} \right\rfloor \quad \leq \quad -2\left\lfloor \frac{e_1 + e_2 + s(e_3 - e_1)}{2s - 1} \right\rfloor + \frac{1}{2}(n - e_1 - e_2 - e_3) - 3. \tag{44}$$

Taking into account the fact that $\frac{1}{2}(n - e_1 - e_2 - e_3)$ is an integer value and using again $\lfloor U_1 \rfloor \leq U_2 \Longleftrightarrow U_1 < U_2 + 1$, the inequality (44) leads to

$$2\left\lfloor \frac{e_1 + e_2 + s(e_3 - e_1)}{2s - 1} \right\rfloor + \frac{-e_3 + s(e_1 - e_2)}{2s - 1} \quad < \quad \frac{1}{2}(n - e_1 - e_2 - e_3) - 2. \tag{45}$$

Or equivalently,

$$\left\lfloor \frac{e_1 + e_2 + s(e_3 - e_1)}{2s - 1} \right\rfloor + \frac{-e_3 + s(e_1 - e_2)}{2(2s - 1)} \quad < \quad \frac{1}{4}(n - e_1 - e_2 - e_3) - 1. \tag{46}$$

Notice that $\lfloor U \rfloor = U - F(U)$, such that $F(U)$ is the fractional part of U. The inequality (46) can be written as

$$\frac{e_1 + e_2 + s(e_3 - e_1)}{2s - 1} + \frac{-e_3 + s(e_1 - e_2)}{2(2s - 1)} \quad < \quad \frac{1}{4}(n - e_1 - e_2 - e_3) + F(U) - 1. \tag{47}$$

Where $U = \dfrac{e_1 + e_2 + s(e_3 - e_1)}{2s - 1} = \dfrac{AB + (1 - s)AC - sBC}{2(2s - 1)}$. Collecting terms leads to (48), which defines the minimal requirement to guarantee the occurrence of the strict Borda paradox.

$$sAB + (1 - s)AC \quad < \quad \frac{2s - 1}{3}(n - 4 + 4F(U)). \tag{48}$$

Finally, if the inequality (48) is reversed, then there is no profile for which the strict Borda paradox occurs.

- **Case 2**: $s < \frac{1}{2}$. In this case, $(2s - 1) < 0$ and the inequalities (36-37) become

$$\frac{e_3 + s(e_2 - e_1)}{1 - 2s} > \beta - \alpha, \qquad \text{and} \qquad \frac{-(e_1 + e_2) + s(e_1 - e_3)}{1 - 2s} > \alpha - \gamma. \tag{49}$$

Or equivalently,

$$\alpha > \beta + \frac{-e_3 + s(e_1 - e_2)}{1 - 2s}, \qquad \text{and} \qquad \gamma > \alpha + \frac{e_1 + e_2 + s(e_3 - e_1)}{1 - 2s}. \tag{50}$$

Notice that $\dfrac{e_1 + e_2 + s(e_3 - e_1)}{1 - 2s}$ always is positive. However, $\dfrac{-e_3 + s(e_1 - e_2)}{1 - 2s}$, which is equivalent to $\dfrac{(1 - s)(AC - AB) + 2sBC}{2(1 - 2s)}$, can be either positive or negative and depends on the sign of $(1 - s)(AC - AB) + 2sBC$. In other words, two cases arise.

- If $(1-s)(AC-AB)+2sBC<0$. In this case, to ensure that a profile can be constructed, it follows that the minimal conditions are where

$$\beta_{min}=0, \tag{51}$$

$$\text{and} \qquad \alpha_{min}=0, \tag{52}$$

$$\text{and} \qquad \gamma_{min}=\left\lfloor \frac{e_1+e_2+s(e_3-e_1)}{1-2s}\right\rfloor +1. \tag{53}$$

Using (51-53) and taking into account that the values of α, β and γ must satisfy $\alpha\geq\alpha_{min}$, $\beta\geq\beta_{min}$, $\gamma\geq\gamma_{min}$ and $\alpha+\beta+\gamma=q=\frac{1}{2}(n-AB)=\frac{1}{2}(n-e_1-e_2-e_3)$, it follows that

$$\left\lfloor \frac{e_1+e_2+s(e_3-e_1)}{1-2s}\right\rfloor +1 \quad \leq \quad \frac{1}{2}(n-e_1-e_2-e_3). \tag{54}$$

Using $\lfloor U_1\rfloor \leq U_2 \Longleftrightarrow U_1<U_2+1$ and collecting terms lead to

$$2(1-s)AB-sBC+(1-s)AC<(1-2s)n. \tag{55}$$

If this inequality is reversed, then there is no profile for which the strict Borda paradox occurs.

- If $(1-s)(AC-AB)+2sBC\geq 0$. In this case,

$$\beta_{min}=0, \tag{56}$$

$$\text{and} \qquad \alpha_{min}=\left\lfloor \frac{-e_3+s(e_1-e_2)}{1-2s}\right\rfloor +1, \tag{57}$$

$$\text{and} \qquad \gamma_{min}=\alpha_{min}+\left\lfloor \frac{e_1+e_2+s(e_3-e_1)}{1-2s}\right\rfloor +1. \tag{58}$$

Using (56-58) and the fact that $\alpha\geq\alpha_{min}$, $\beta\geq\beta_{min}$, $\gamma\geq\gamma_{min}$ and $\alpha+\beta+\gamma=q=\frac{1}{2}(n-AB)=\frac{1}{2}(n-e_1-e_2-e_3)$, it follows

$$2\left\lfloor \frac{-e_3+s(e_1-e_2)}{1-2s}\right\rfloor + \left\lfloor \frac{e_1+e_2+s(e_3-e_1)}{1-2s}\right\rfloor +3 \leq \frac{1}{2}(n-e_1-e_2-e_3). \tag{59}$$

Or equivalently,

$$\left\lfloor \frac{e_1+e_2+s(e_3-e_1)}{1-2s}\right\rfloor \leq -2\left\lfloor \frac{-e_3+s(e_1-e_2)}{1-2s}\right\rfloor + \frac{1}{2}(n-e_1-e_2-e_3)-3. \tag{60}$$

Using $\lfloor U_1\rfloor \leq U_2 \Longleftrightarrow U_1<U_2+1$, the inequality (60) is equivalent to

$$2\left\lfloor \frac{-e_3+s(e_1-e_2)}{1-2s}\right\rfloor + \frac{e_1+e_2+s(e_3-e_1)}{1-2s} < \frac{1}{2}(n-e_1-e_2-e_3)-2. \tag{61}$$

Or equivalently,

$$\left\lfloor \frac{-e_3+s(e_1-e_2)}{1-2s}\right\rfloor + \frac{e_1+e_2+s(e_3-e_1)}{2(1-2s)} < \frac{1}{4}(n-e_1-e_2-e_3)-1 \tag{62}$$

Using $\lfloor U\rfloor = U-F(U)$ and collecting terms lead to

$$(1-s)AC+sBC \quad < \quad \frac{1-2s}{3}(n-4+4F(U)) \tag{63}$$

Such that $U=\dfrac{-e_3+s(e_1-e_2)}{1-2s}=\dfrac{(1-s)(AC-AB)+2sBC}{2(1-2s)}$. Finally, if the inequality (63) is reversed, then there is no profile for which the strict Borda paradox occurs. *Q.E.D.*

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